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A restricted Epstein zeta function and the evaluation of some definite integrals

by

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1. Introduction. A nonzero integer d is called a *discriminant* if $d \equiv 0$ or $1 \pmod 4$. We set

(1) $$d = \Delta(d) f(d)^2,$$

where $f(d)$ is the largest positive integer for which $\Delta(d) = d/f(d)^2$ is a discriminant. The integer $f(d)$ is called the *conductor* of the discriminant d. The discriminant d is called *fundamental* if $f(d) = 1$. A discriminant d is fundamental if and only if d is odd and squarefree or d is even, $d/4$ is squarefree and $d/4 \equiv 2$ or $3 \pmod 4$. We note that the discriminant $\Delta(d)$ is fundamental so that $f(\Delta(d)) = 1$ and $\Delta(\Delta(d)) = \Delta(d)$. If $d = \Delta' {f'}^2$, where Δ' is a fundamental discriminant and f' is a positive integer, then $\Delta' = \Delta(d)$ and $f' = f(d)$. The discriminant $\Delta(d)$ is called the fundamental discriminant associated with the discriminant d. If d_1 and d_2 are discriminants then $d_1 d_2$ is also a discriminant. We have

$$d_1 = \Delta(d_1) f(d_1)^2, \quad d_2 = \Delta(d_2) f(d_2)^2, \quad d_1 d_2 = \Delta(d_1)\Delta(d_2)(f(d_1) f(d_2))^2$$

so that

$$d_1 d_2 = \Delta(\Delta(d_1)\Delta(d_2))(f(\Delta(d_1)\Delta(d_2)) f(d_1) f(d_2))^2,$$

and thus

$$\Delta(d_1 d_2) = \Delta(\Delta(d_1)\Delta(d_2)), \qquad f(d_1 d_2) = f(\Delta(d_1)\Delta(d_2)) f(d_1) f(d_2).$$

In particular we have

$$f(d_1) \mid f(d_1 d_2), \qquad f(d_2) \mid f(d_1 d_2).$$

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If k is a positive integer then k^2 is a discriminant with $\Delta(k^2) = 1$, $f(k^2) = k$, so that for any discriminant d we have

$$\Delta(dk^2) = \Delta(\Delta(d)\Delta(k^2)) = \Delta(\Delta(d)) = \Delta(d),$$
$$f(dk^2) = f(\Delta(d)\Delta(k^2))f(d)f(k^2) = f(\Delta(d))f(d)k = f(d)k.$$

When there is no confusion, we write $\Delta = \Delta(d)$ and $f = f(d)$.

Throughout the rest of this paper, d represents a nonsquare discriminant and n represents a positive integer. For integers a, b and c with $\gcd(a, b, c) = 1$, we use (a, b, c) to denote the primitive, integral, binary quadratic form $ax^2 + bxy + cy^2$. A form (a, b, c) with $b^2 - 4ac = d$ is called a *form of discriminant* d. Such a form is irreducible in $\mathbb{Z}[x, y]$ as d is not a square. Two forms (a, b, c) and (a', b', c') are *equivalent* if and only if there exist integers r, s, t and u with $ru - st = 1$ such that the substitution $x = rX + sY, y = tX + uY$ transforms (a, b, c) to (a', b', c'). If (a, b, c) is equivalent to (a', b', c'), we write $(a, b, c) \sim (a', b', c')$. The relation $\sim$ is an equivalence relation on the set of forms of discriminant d. We denote the class of (a, b, c) by $[a, b, c]$. The classes of primitive, integral, binary quadratic forms of discriminant d (only positive-definite forms are used if $d < 0$) form a finite abelian group under Gaussian composition (see for example [1: Chapter 4]). We denote this group by $H(d)$ and its order by $h(d)$. The cosets of the subgroup of squares in $H(d)$ are called *genera* and we denote the group of genera by $G(d)$. The identity element of $G(d)$ is called the *principal genus*. By group theory we have $|G(d)| = 2^t$, where $t = t(d)$ is a nonnegative integer. The value of $t(d)$ is given by [7: §153, pp. 409–413; §151, pp. 400–407] (see also [13: p. 277])

$$t(d) = \begin{cases} \omega(d) & \text{if } d \equiv 0 \pmod{32}, \\ \omega(d) - 2 & \text{if } d \equiv 4 \pmod{16}, \\ \omega(d) - 1 & \text{otherwise,} \end{cases}$$

where $\omega(d)$ denotes the number of distinct prime factors of d. Thus $|G| = h(d)/2^t$ for any $G \in G(d)$.

Let $[a, b, c] \in H(d)$. The positive integer n is said to be *represented by the form* (a, b, c) if there exist integers x and y with $ax^2 + bxy + cy^2 = n$, and the pair (x, y) is called a representation. If $d < 0$ every representation (x, y) is called *primary*. If $d > 0$ the representation (x, y) is called *primary* if it satisfies

$$(2) \qquad 2ax + (b - \sqrt{d})y > 0 \quad \text{and} \quad 1 \le \left| \frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y} \right| < \varepsilon^2,$$

where

$$(3) \qquad \varepsilon = \varepsilon(d) = (x_0 + y_0\sqrt{d})/2,$$

and $(x_0, y_0) = (x_0(d), y_0(d))$ is the solution in positive integers to the equation $x^2 - dy^2 = 4$ for which y_0 is least (see for example [12: p. 282]). We set

$$R_{(a,b,c)}(n,d) = \text{card}\{(x,y) \in \mathbb{Z}^2 : ax^2+bxy+cy^2 = n,\ (x,y) \text{ primary}\}. \tag{4}$$

$R_{(a,b,c)}(n,d)$ is finite and $R_{(a,b,c)}(n,d) = R_{(a',b',c')}(n,d)$ if $(a,b,c) \sim (a',b',c')$ (see for example [12: §11.4]). Thus we can define

$$R_{[a,b,c]}(n,d) = R_{(a,b,c)}(n,d). \tag{5}$$

For $G \in G(d)$, we set

$$R_G(n,d) = \sum_{K \in G} R_K(n,d). \tag{6}$$

When $d < 0$, Huard, Kaplan and Williams [13: Theorem 8.1] have obtained an explicit formula for $R_G(n,d)$. Using this formula they showed [13: Theorem 10.2] that for $s \to 1^+$,

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{h(d)}{2^{t(d)}} \cdot \frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + B_G(d) + O(s-1), \tag{7}$$

where $B_G(d)$ is an explicit constant depending on d and G. In this paper, we extend their ideas to the case $d > 0$. In Section 2 we obtain a formula for $R_G(n,d)$ when $d > 0$ (see Theorem 1). In Section 4 we use this formula to determine $\sum_{n=1}^{\infty} R_G(n,d)/n^s$ for $d > 0$ and $s > 1$ (see Theorem 3). From Theorem 3 we deduce that

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{h(d)}{2^{t(d)}} \cdot \frac{\log \varepsilon(d)}{\sqrt{d}} \cdot \frac{1}{s-1} + B(d) + \beta(d,G) + O(s-1), \tag{8}$$

where $B(d)$ is a constant depending only on d and not on G and $\beta(d,G)$ is an explicit constant depending on both d and G (see Theorem 4).

If $Q = (a,b,c)$ is a positive-definite binary quadratic form of discriminant $d < 0$, the *Epstein zeta function* $Z_Q(s)$ corresponding to Q is defined for $s > 1$ by the infinite series

$$Z_Q(s) = \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0)}}^{\infty} \frac{1}{Q(x,y)^s} \tag{9}$$

(see for example [5], [8], [14], [15]). The behaviour of $Z_Q(s)$ near $s = 1$ is given by Kronecker's limit formula (see for example [13: p. 300], [15: p. 14])

$$Z_Q(s) = \frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + K(a,b,c) + O(s-1), \tag{10}$$

where $K(a,b,c)$ is an explicit constant depending only on a, b and c. Let

$G \in G(d)$. As

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \sum_{[a,b,c]\in G} Z_{(a,b,c)}(s)$$

we obtain

$$\frac{h(d)}{2^{t(d)}} \cdot \frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + B_G(d) + O(s-1) = \sum_{[a,b,c]\in G} \left(\frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + K(a,b,c) + O(s-1) \right),$$

so that

$$B_G(d) = \sum_{[a,b,c]\in G} K(a,b,c). \tag{11}$$

The Chowla–Selberg formula for genera, which was proved by Huard, Kaplan and Williams [13: Theorem 1.1] in 1995, is obtained by putting the explicit values of $B_G(d)$ and $K(a,b,c)$ into (11) and exponentiating the resulting formula.

We now define an analogue of the Epstein zeta function (9) in the case of an indefinite binary quadratic form $Q = (a,b,c)$ of discriminant $d > 0$ with $a > 0$. We remark that if the form (a,b,c) is indefinite, then we can always replace it by an equivalent one with $a > 0$. To see this, recall that an indefinite form (a,b,c) represents both positive and negative integers. Let k be a positive integer represented by (a,b,c). Then there is a positive integer l dividing k which is properly represented by (a,b,c). Hence $(a,b,c) \sim (l,b',c')$ for some integers b' and c'. We call our analogue of (9) the *restricted Epstein zeta function* and denote it by $Z_Q(s)$. We set

$$Z_Q(s) = \sum_{\substack{x,y=-\infty \\ Q(x,y)>0 \\ 2ax+(b-\sqrt{d})y>0 \\ 1\le\left|\frac{2ax+(b+\sqrt{d})y}{2ax+(b-\sqrt{d})y}\right|<\varepsilon^2}}^{\infty} \frac{1}{Q(x,y)^s}. \tag{12}$$

It is shown in Section 3 that the series in (12) converges for $s > 1$ so that $Z_Q(s)$ is defined for $s > 1$. Also in Section 3, it is shown that as $s \to 1^+$

$$Z_Q(s) = \frac{\log \varepsilon(d)}{\sqrt{d}} \cdot \frac{1}{s-1} + C_Q + O(s-1) \tag{13}$$

for an explicit constant C_Q (see Theorem 2). We remark that Zagier [16: Theorems, pp. 166–167] has considered a different analogue of the Epstein zeta function, namely, the infinite series

$$\sum_{x=1,\, y=0}^{\infty} \frac{1}{Q(x,y)^s}$$

for an indefinite binary quadratic form $Q = (a,b,c)$ of discriminant $d > 0$ with $a > 0$, $b > 0$ and $c > 0$. Let $G \in G(d)$. Since

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \sum_{[Q]\in G} Z_Q(s), \tag{14}$$

we obtain from (8), (13) and (14)

$$\begin{aligned} \frac{h(d)\log\varepsilon(d)}{2^{t(d)}\sqrt{d}} \cdot \frac{1}{s-1} &+ B(d) + \beta(d,G) + O(s-1) \\ &= \sum_{[Q]\in G} \left(\frac{\log\varepsilon(d)}{\sqrt{d}} \cdot \frac{1}{s-1} + C_Q + O(s-1) \right) \\ &= \frac{h(d)\log\varepsilon(d)}{2^{t(d)}\sqrt{d}} \cdot \frac{1}{s-1} + \sum_{[Q]\in G} C_Q + O(s-1), \end{aligned}$$

so that

$$B(d) + \beta(d,G) = \sum_{[Q]\in G} C_Q. \tag{15}$$

The formula (15) provides an analogue of the Chowla–Selberg formula for genera in the case of positive discriminants. However the constant $B(d)$ contains the quantity $L'(1,\Delta)$ (see (98) and (100)), which is difficult to give explicitly (see Deninger [6]). Thus in Section 5 we eliminate $B(d)$ from (15) to obtain a simpler formula. Let G_1 and G_2 be two genera of $G(d)$. Then, from (15), we obtain

$$\beta(d,G_1) - \beta(d,G_2) = \sum_{[Q]\in G_1} C_Q - \sum_{[Q]\in G_2} C_Q. \tag{16}$$

Putting the explicit expressions for $\beta(d,G_k)$ $(k = 1,2)$ and C_Q into (16), we obtain Theorem 5.

By taking particular choices of the genera G_1 and G_2 in Theorem 5, we are able to evaluate explicitly certain definite integrals. The nature of these integrals suggests that it would be difficult to evaluate them by conventional means, a view previously expressed by Chowla ([3: p. 372], [4: p. 1019]). These integrals are given in Theorems 6–10. They include three integrals given by Herglotz [11: p. 14] as well as many new ones such as

$$\int_0^1 \frac{\tan^{-1}(t^{3+\sqrt{8}})}{1+t^2}\,dt = \frac{1}{16}\log 2\,\log(3+\sqrt{8}) \tag{17}$$

and

$$\begin{aligned}\int_0^1 \frac{\log(1+t^{13+\sqrt{168}})}{1+t}\,dt = {} & \frac{\pi^2}{24}(13-\sqrt{672}) \\ & + \frac{1}{2}\log(1+\sqrt{2})\log\left(\frac{5+\sqrt{21}}{2}\right) \\ & + \frac{1}{4}\log(2+\sqrt{3})\log(15+\sqrt{224}) \\ & + \frac{1}{4}\log(5+\sqrt{24})\log(8+\sqrt{63}) \\ & + \frac{1}{2}\log 2\,\log(2(13+\sqrt{168})^{3/2})\end{aligned} \tag{18}$$

(see Theorem 10).

2. Formula for $R_G(n,d)$. Proof of Theorem 1. In this section, up to and including Lemma 12, d may be either positive or negative. From Lemma 13 on, and throughout the rest of the paper, d is assumed to be positive.

The discriminants $-4, 8, -8$ and $p^* = (-1)^{(p-1)/2}p$ (p prime > 2), are called *prime discriminants*. The prime discriminants corresponding to the discriminant d are the discriminants $p_1^*, \ldots, p_{t+1}^*$, together with p_{t+2}^* if $d \equiv 0 \pmod{32}$, where $t = t(d)$, given as follows:

- $d \equiv 1 \pmod 4$ or $d \equiv 4 \pmod{16}$,
  $p_1 < \ldots < p_{t+1}$ are the odd prime divisors of d.
- $d \equiv 12 \pmod{16}$ or $d \equiv 16 \pmod{32}$,
  $p_1 < \ldots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = -4$.
- $d \equiv 8 \pmod{32}$,
  $p_1 < \ldots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = 8$.
- $d \equiv 24 \pmod{32}$,
  $p_1 < \ldots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = -8$.
- $d \equiv 0 \pmod{32}$,
  $p_1 < \ldots < p_{t-1}$ are the odd prime divisors of d, $p_t^* = -4$, $p_{t+1}^* = 8$, $p_{t+2}^* = -8$.

Following Huard, Kaplan and Williams [13] we denote the set of prime discriminants corresponding to d by $P(d)$. We denote the set of all products of pairwise coprime elements of $P(d)$ by $F(d)$.

It is known that a fundamental discriminant d can be written uniquely as a product of pairwise coprime prime discriminants and that any such product is a fundamental discriminant [15: Proposition 9]. It is easy to check that the prime discriminants occurring in such a decomposition are precisely the elements of $P(d)$. It is convenient at this point to note some properties of the set $F(d)$.

LEMMA 1. (a) $F(d) = \{d_1 : d_1$ *is a fundamental discriminant*, $d_1 \mid d$, *and* d/d_1 *is a discriminant*$\}$.

(b) *For any positive integer* k, $P(d) \subseteq P(dk^2)$ *and* $F(d) \subseteq F(dk^2)$. *Also*, $P(\Delta) \subseteq P(d)$, $1 \in F(d)$, $\Delta \in F(d)$, $|F(d)| = 2^{t(d)+1}$, *and*

$$|P(d)| = \begin{cases} t(d)+2 & \text{if } d \equiv 0 \pmod{32}, \\ t(d)+1 & \text{otherwise.} \end{cases}$$

(c) *If* $d_1 \in F(d)$ *then* $f(d/d_1) \mid f(d)$.

(d) *Let* m *be a positive integer such that* $m \mid f$. *Let* $d_1 \in F(d/m^2)$. *Then*

$$f(d/m^2 d_1) \mid f/m \quad \text{and} \quad m \mid f(d/d_1).$$

(e) *Let* m *be a positive integer. Then*

$$m \mid f(d),\ d_1 \in F(d/m^2) \ \Leftrightarrow\ d_1 \in F(d),\ m \mid f(d/d_1).$$

Proof. (a), (b). These two parts of the lemma are given in [13: Lemma 2.1] for the case $d < 0$. It is easy to check that they are also valid for $d > 0$.

(c) As $d_1 \in F(d)$, by (a) we see that d_1 and d/d_1 are discriminants with $d_1 \cdot d/d_1 = d$, so that by the properties given at the beginning of Section 1, we have $f(d/d_1) \mid f(d)$.

(d) As m is a positive integer such that $m \mid f$ we have $m^2 \mid d$, d/m^2 is a discriminant, and $f(d/m^2) = f/m$. Further, as $d_1 \in F(d/m^2)$, d_1 is a fundamental discriminant such that $d_1 \mid d/m^2$ and $d_2 = \frac{d/m^2}{d_1}$ is a discriminant. From $d_1 d_2 = d/m^2$ we have $f(d_2) \mid f(d/m^2)$, that is, $f(d/m^2 d_1) \mid f/m$, as asserted. Also $d/d_1 = d_2 m^2$ so that

$$f(d/d_1) = f(d_2 m^2) = f(d_2)m,$$

that is, $m \mid f(d/d_1)$.

(e) Suppose first that $m \mid f(d)$ and $d_1 \in F(d/m^2)$. Then $d_1 \in F(d)$ by (b) and $m \mid f(d/d_1)$ by (d). Hence we have shown that

$$m \mid f(d),\ d_1 \in F(d/m^2) \ \Rightarrow\ d_1 \in F(d),\ m \mid f(d/d_1).$$

Now suppose that $d_1 \in F(d)$ and $m \mid f(d/d_1)$. By (a), d_1 is a fundamental discriminant such that $d_1 \mid d$ and d/d_1 is a discriminant. As $m \mid f(d/d_1)$, from

$$d/d_1 = \Delta(d/d_1) f(d/d_1)^2,$$

we deduce that $d_1 \mid d/m^2$ and d/d_1m^2 is a discriminant so that $d_1 \in F(d/m^2)$ by (a). As $m \mid f(d/d_1)$ we have $m \mid f(d)$ by (c) so we have shown that

$$d_1 \in F(d),\, m \mid f(d/d_1) \;\Rightarrow\; m \mid f(d),\, d_1 \in F(d/m^2).$$

This completes the proof of Lemma 1. ∎

Next, we recall the basic properties of generic characters (see for example [1: Chapter 4]). Let $p^* \in P(d)$ and $K \in H(d)$. For any positive integer k coprime with p^*, which is represented by K, it is known that $\left(\frac{p^*}{k}\right)$ has the same value, so we can set

$$\gamma_{p^*}(K) = \left(\frac{p^*}{k}\right) = \pm 1. \tag{19}$$

Let $G \in G(d)$. It is known that for any $K \in G$, $\gamma_{p^*}(K)$ has the same value, so we can set $\gamma_{p^*}(G) = \gamma_{p^*}(K)$. Also,

$$\gamma_{p^*}(G_1G_2) = \gamma_{p^*}(G_1)\gamma_{p^*}(G_2), \tag{20}$$

for $G_1, G_2 \in G(d)$. An important result of genus theory is the following product formula due to Gauss (see for example [9: equation (9)]).

Lemma 2. (a) *If* $G \in G(d)$ *then with* $\Delta = \Delta(d)$,

$$\prod_{p^* \in P(\Delta)} \gamma_{p^*}(G) = 1, \tag{21}$$

together with

$$\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1 \quad \text{if } d \equiv 0 \pmod{32}. \tag{22}$$

(b) *Moreover, if* $\delta_{p^*} = \pm 1$ *for each* $p^* \in P(d)$ *and* $\prod_{p^* \in P(\Delta)} \delta_{p^*} = 1$, *together with*

$$\delta_{-4}\delta_8\delta_{-8} = 1 \quad \text{if } d \equiv 0 \pmod{32},$$

then there exists a unique genus $G \in G(d)$ *with*

$$\gamma_{p^*}(G) = \delta_{p^*} \quad \text{for each } p^* \in P(d).$$

For $d_1 \in F(d)$, *we set*

$$\gamma_{d_1}(G) = \prod_{p^* \in P(d_1)} \gamma_{p^*}(G) = \pm 1. \tag{23}$$

We let $v_p(n)$ denote the exponent of the highest power of the prime p dividing n. Following [13] we define for all discriminants d the derived genus $G_m \in G(d/(m,f)^2)$ of $G \in G(d)$, where m is a positive integer all of whose prime factors p divide d and satisfy

$$p \nmid \Delta \;\Rightarrow\; v_p(m) \le v_p(f). \tag{24}$$

We begin with the case when m is a prime.

LEMMA 3. *Let p be a prime with $p \mid d$, and let $G \in G(d)$. Then there is a unique genus*

$$G_p \in \begin{cases} G(d/p^2) & \text{if } p \mid f, \\ G(d) & \text{if } p \nmid f, \end{cases} \tag{25}$$

such that in the case $p \mid f$,

$$\gamma_{q^*}(G_p) = \gamma_{q^*}(G) \quad \text{for all } q^* \in P(d/p^2), \tag{26}$$

and in the case $p \nmid f$, for every $q^* \in P(d)$ with $p \nmid q^*$,

$$\gamma_{q^*}(G_p) = \left(\frac{q^*}{p}\right)\gamma_{q^*}(G),$$

and, *for the unique $q^* \in P(d)$ with $p \mid q^*$,*

$$\gamma_{q^*}(G_p) = \left(\frac{d/q^*}{p}\right)\gamma_{q^*}(G) = \left(\frac{\Delta/q^*}{p}\right)\gamma_{q^*}(G).$$

Proof. The proof is exactly the same as the proof of Proposition 3.1 in [13] for the case $d < 0$. ■

Next, we define G_{p^i} for $p \mid d$ and $i \geq 0$. We set $G_1 = G$. By (25), we define successively

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2i}) \quad \text{for } i = 1, \ldots, v_p(f). \tag{27}$$

If in addition $p \mid \Delta$, as $p \nmid f/p^{v_p(f)}$, we define successively

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2v_p(f)}), \quad i = 1 + v_p(f), \ldots \tag{28}$$

Thus, for any $p \mid d$, we have defined $G_{p^i} \in G(d/(p^i, f)^2)$ for any $i \geq 0$ if $p \mid \Delta$ and for $0 \leq i \leq v_p(f)$ if $p \nmid \Delta$. For $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ satisfying (24), we define

$$G_m = (\ldots((G_{p_1^{\alpha_1}})_{p_2^{\alpha_2}})\ldots)_{p_r^{\alpha_r}} \in G(d/(m, f)^2). \tag{29}$$

It is easily checked that the order of the p_i's does not matter.

LEMMA 4. (a) *Let p be a prime with $p \mid d$. Let $d_1 \in F(d/(p, f)^2)$. Then, for any $G \in G(d)$, we have*

$$\gamma_{d_1}(G_p) = \begin{cases} \gamma_{d_1}(G) & \text{if } p \mid f, \\ \left(\frac{d_1}{p}\right)\gamma_{d_1}(G) & \text{if } p \nmid f,\ p \nmid d_1, \\ \left(\frac{d/d_1}{p}\right)\gamma_{d_1}(G) & \text{if } p \nmid f,\ p \mid d_1. \end{cases}$$

(b) *If m is a positive integer with $m \mid f$, $G \in G(d)$ and $d_1 \in F(d/m^2)$ then $\gamma_{d_1}(G_m) = \gamma_{d_1}(G)$.*

Proof. The proof is exactly the same as the proof of Lemma 3.1 in [13] for the case $d < 0$. ■

Following [13] we define a prime p to be a *null prime* relative to n and d if

$$v_p(n) \equiv 1 \pmod 2, \qquad v_p(n) < 2v_p(f). \tag{30}$$

We denote the set of all such null primes by $\mathrm{Null}(n,d)$.

LEMMA 5. *If* $\mathrm{Null}(n,d) \neq \emptyset$, *then* $R_K(n,d) = 0$ *for each* $K \in H(d)$.

Proof. The proof is exactly the same as the proof of Proposition 4.1 in [13] for the case $d < 0$. ■

Next, as in [13: Section 4], we introduce three positive integers M, U and Q:

(31) $M = M(n,d)$ is the largest integer such that $M^2 \mid n$ and $M \mid f$,

$$U = U(n,d) = \prod_{\substack{p \mid d \\ p \nmid f}} p^{v_p(n)}, \tag{32}$$

$$Q = Q(n,d) = U(n/M^2, d/M^2) = \prod_{\substack{p \mid d/M^2 \\ p \nmid f/M}} p^{v_p(n/M^2)}. \tag{33}$$

LEMMA 6. (a) *If* $\mathrm{Null}(n,d) = \emptyset$ *then*

$$(n/M^2, f/M) = 1 \quad \textit{and} \quad (n/M^2Q, d/M^2) = 1.$$

(b) $(n,f) = 1$ *if and only if* $\mathrm{Null}(n,d) = \emptyset$ *and* $M = 1$.

Proof. The proof is the same as the proof of Lemma 4.1 in [13]. ■

For $d_1 \in F(d)$ and $(n,f) = 1$, we set

$$S(n, d_1, d/d_1) = \sum_{\mu\nu = n} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right), \tag{34}$$

where μ and ν run through all positive integers with $\mu\nu = n$.

LEMMA 7. *Let* $(n,f) = 1$ *and let* p *be a prime dividing both* n *and* d. *Then, for* $G \in G(d)$, *we have*

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1 \in F(d)} \gamma_{d_1}(G_p) S(n/p, d_1, d/d_1).$$

Proof. The proof is the same as that of Lemma 5.1 in [13]. ■

LEMMA 8. *Let* $(n,f) = 1$. *Then, for* $G \in G(d)$, *we have*

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1 \in F(d)} \gamma_{d_1}(G_U) S(n/U, d_1, d/d_1),$$

where U *is defined in* (32).

Proof. This follows by repeatedly applying Lemma 7 to all the primes dividing the integer U. ■

LEMMA 9. *Let p be a prime with $p\,|\,d$, $p\nmid f$. Let $K\in H(d)$. Then*

(a) *K contains a form (a,b,cp) with $p\nmid ac$, $p\,|\,b$;*

(b) *the mapping $\phi_p : H(d)\to H(d)$ given by $\phi_p([a,b,cp]) = [ap,b,c]$, where (a,b,cp) is as in* (a), *is a bijection*;

(c) *if $G\in G(d)$ and $K\in G$ then $\phi_p(K)\in G_p$.*

Proof. The proof is the same as the proof of Lemma 7.1 in [13]. ■

LEMMA 10. *Let p be a prime with $p\,|\,n$, $p\,|\,d$ and $p\nmid f$. Then, for $K\in H(d)$, we have $R_K(n,d) = R_{\phi_p(K)}(n/p,d)$.*

Proof. Let $(a,b,cp)\in K$ with $p\nmid ac$, $p\,|\,b$. Then $(ap,b,c)\in\phi_p(K)$. We set

$$S = \{(x,y)\in\mathbb{Z}^2 : ax^2+bxy+cpy^2 = n,\ (x,y)\text{ primary}\},$$
$$T = \{(X,Y)\in\mathbb{Z}^2 : apX^2+bXY+cY^2 = n/p,\ (X,Y)\text{ primary}\}.$$

It is easily checked that $(X,Y)\mapsto(pX,Y)$ is a bijection from T to S. ■

LEMMA 11. *Let p be a prime with $p\,|\,n$, $p\,|\,d$ and $p\nmid f$. Then, for $G\in G(d)$, we have $R_G(n,d) = R_{G_p}(n/p,d)$.*

Proof. We have

$$\begin{aligned} R_G(n,d) &= \sum_{K\in G} R_K(n,d) = \sum_{K\in G} R_{\phi_p(K)}(n/p,d)\\ &= \sum_{K'\in G_p} R_{K'}(n/p,d) = R_{G_p}(n/p,d), \end{aligned}$$

by Lemmas 9 and 10. ■

We are now ready to prove our first reduction formula.

PROPOSITION 1. *For $G\in G(d)$, we have*

$$R_G(n,d) = R_{G_U}(n/U,d)$$

where $U=U(n,d)$ is defined in (32).

Proof. This follows from Lemma 11 by repeatedly applying it to all the primes dividing U. ■

LEMMA 12. *Let $p\,|\,f$, $K\in H(d)$ and let l be a positive integer. Then*

(a) *K contains a form (a,b,c) with $p\,|\,b$, $p^2\,|\,c$ and $(a,pl)=1$;*

(b) *the mapping $\theta_p : H(d)\to H(d/p^2)$ given by $\theta_p([a,b,c]) = [a,b/p,c/p^2]$, where (a,b,c) is as in* (a), *is a surjective homomorphism*;

(c) *if $G\in G(d)$ and $K\in G$ then $\theta_p(K)\in G_p$;*

(d) *the mapping $\widehat{\theta}_p : G(d)\to G(d/p^2)$ given by $\widehat{\theta}_p(G) = G_p$ is a surjective homomorphism.*

Proof. The proof is exactly the same as that of Lemma 6.1 in [13]. ∎

From this point on we assume that $d > 0$.

LEMMA 13. *Let* $d > 0$, $[a, b, c] \in H(d)$ *and let* m *be a positive integer. Set*

$$S = \Big\{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n,\ 2ax + (b - \sqrt{d})y > 0,\ 1 \le \left|\frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y}\right| < \varepsilon^2\Big\},$$

$$T = \Big\{(X, Y) \in \mathbb{Z}^2 : aX^2 + bXY + cY^2 = n,\ 2aX + (b - \sqrt{d})Y > 0,\ \varepsilon^{2m} \le \left|\frac{2aX + (b + \sqrt{d})Y}{2aX + (b - \sqrt{d})Y}\right| < \varepsilon^{2m+2}\Big\},$$

where $\varepsilon = \varepsilon(d)$ *is defined in* (3). *Then* $\operatorname{card} S = \operatorname{card} T$.

Proof. Let

$$\varepsilon' = \frac{1}{\varepsilon} = \frac{x_0 - y_0\sqrt{d}}{2} \quad \text{and} \quad \varepsilon^m = \frac{t + u\sqrt{d}}{2},$$

where t and u are rational numbers. Then

$$\varepsilon'^m = \frac{t - u\sqrt{d}}{2}.$$

Adding we obtain $t = \varepsilon^m + \varepsilon'^m$. As ε is an algebraic integer, so are $\varepsilon', \varepsilon^m$ and ε'^m. Hence t is an algebraic integer and thus, as it is rational, it must be an integer. Similarly

$$u = y_0 \frac{\varepsilon^m - \varepsilon'^m}{\varepsilon - \varepsilon'}$$

is an algebraic integer, and thus as it is rational, it must be an integer. Finally, as $\varepsilon\varepsilon' = 1$, we deduce that the integers t and u satisfy $t^2 - du^2 = 4$.

We define a map from S to T by $(x, y) \mapsto (X, Y)$, where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (t - bu)/2 & -cu \\ au & (t + bu)/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Easy calculations show that

$$ax^2 + bxy + cy^2 = aX^2 + bXY + cY^2,$$
$$2aX + (b + \sqrt{d})Y = \varepsilon^m(2ax + (b + \sqrt{d})y).$$

Hence

$$2aX + (b - \sqrt{d})Y = \varepsilon'^m(2ax + (b - \sqrt{d})y).$$

It is now easily verified that the map $(x, y) \mapsto (X, Y)$ is a bijection. ∎

LEMMA 14. *Let $d > 0$. Let p be a prime with $p \mid M$, where M is defined in* (31). *Then, for any $K \in H(d)$, we have*

$$R_K(n,d) = \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} R_{\theta_p(K)}(n/p^2, d/p^2).$$

Proof. We begin by choosing $(a,b,c) \in K$ with $p \nmid a$, $p \mid b$ and $p^2 \mid c$ so that $\theta_p(K) = [a, b/p, c/p^2]$. Then we set

$$S = \Big\{ (x,y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n,\ 2ax + (b - \sqrt{d})y > 0,$$
$$1 \le \left| \frac{2ax + (b+\sqrt{d})y}{2ax + (b-\sqrt{d})y} \right| < \varepsilon(d)^2 \Big\},$$

$$T = \Big\{ (X,Y) \in \mathbb{Z}^2 : \frac{n}{p^2} = aX^2 + \frac{bXY}{p} + \frac{cY^2}{p^2}, 2aX + (b - \sqrt{d})\,\frac{Y}{p} > 0,$$
$$1 \le \left| \frac{2aX + (b+\sqrt{d})Y/p}{2aX + (b-\sqrt{d})Y/p} \right| < \varepsilon(d)^2 \Big\},$$

$$V = \Big\{ (X,Y) \in \mathbb{Z}^2 : \frac{n}{p^2} = aX^2 + \frac{bXY}{p} + \frac{cY^2}{p^2}, 2aX + (b - \sqrt{d})\,\frac{Y}{p} > 0,$$
$$1 \le \left| \frac{2aX + (b+\sqrt{d})Y/p}{2aX + (b-\sqrt{d})Y/p} \right| < \varepsilon(d/p^2)^2 \Big\}.$$

All solutions in integers to $x^2 - dy^2 = 4$ are given by

$$\frac{x + y\sqrt{d}}{2} = \pm \varepsilon^m, \qquad m \in \mathbb{Z},$$

(see for example [12: Theorem 4.4, p. 281]). As $x = x_0$, $y = py_0$ is an integral solution of $x^2 - (d/p^2)y^2 = 4$, we have

$$\varepsilon(d) = \frac{x_0 + y_0\sqrt{d}}{2} = \frac{x + (y/p)\sqrt{d}}{2} = \pm \varepsilon(d/p^2)^m$$

for some $m \in \mathbb{Z}$. Moreover as $\varepsilon(d)$ and $\varepsilon(d/p^2)$ are both > 1 we have $\varepsilon(d) = \varepsilon(d/p^2)^m$ and m is a positive integer. The map from T to S given by $(X,Y) \mapsto (pX, Y)$ is easily seen to be a bijection. Thus

$$R_K(n,d) = \operatorname{card} S = \operatorname{card} T = m \operatorname{card} V = \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} R_{\theta_p(K)}(n/p^2, d/p^2),$$

by Lemma 13. ∎

Our next lemma is the analogue of [13: Lemma 6.3] for the case $d > 0$. As the proof in [13] is fairly brief, we provide all the details here.

LEMMA 15. *Let $d > 0$. Let p be a prime with $p \mid M$. Then, for $G \in G(d)$, we have*

$$R_G(n,d) = \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \cdot \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} R_{G_p}(n/p^2, d/p^2).$$

Proof. Let $G \in G(d)$ and $L \in G_p$. As $G \in G(d) = H(d)/H^2(d)$ there exists a class $K_1 \in H(d)$ such that $G = K_1 H^2(d)$. Thus $K_1 \in G$ and so $\theta_p(K_1) \in G_p$. Hence

$$G_p = \theta_p(K_1) H^2(d/p^2).$$

As $L \in G_p$ there exists $L_1 \in H(d/p^2)$ such that $L = \theta_p(K_1) L_1^2$. Further, as the homomorphism $\theta_p : H(d) \to H(d/p^2)$ is surjective, there exists a class $K_2 \in H(d)$ such that $\theta_p(K_2) = L_1$. Set $A = K_1 K_2^2$ so that $A \in G$ and

$$\theta_p(A) = \theta_p(K_1 K_2^2) = \theta_p(K_1)\theta_p(K_2)^2 = \theta_p(K_1) L_1^2 = L.$$

Also $G = AH^2(d)$. Set

$$N_G(L) = \sum_{\substack{K \in G \\ \theta_p(K) = L}} 1.$$

Then

$$\begin{aligned} N_G(L) &= |\{K \in G : \theta_p(K) = L\}| = |\{K \in H(d) : \theta_p(K) = L\} \cap G| \\ &= |A \ker \theta_p \cap G| = |A \ker \theta_p \cap AH^2(d)| \\ &= |A(\ker \theta_p \cap H^2(d))| = |\ker \theta_p \cap H^2(d)|, \end{aligned}$$

so that $N_G(L)$ is independent of G and L. Hence

$$\begin{aligned} |G| &= \sum_{K \in G} 1 = \sum_{\substack{K \in G \\ \theta_p(K) \in G_p}} 1 = \sum_{L \in G_p} \sum_{\substack{K \in G \\ \theta_p(K) = L}} 1 \\ &= \sum_{L \in G_p} N_G(L) = N_G(L) \sum_{L \in G_p} 1 = N_G(L)|G_p|, \end{aligned}$$

so that

$$N_G(L) = \frac{|G|}{|G_p|} = \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}}.$$

Hence we have

$$\begin{aligned} R_G(n,d) &= \sum_{K \in G} R_K(n,d) \\ &= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \sum_{K \in G} R_{\theta_p(K)}(n/p^2, d/p^2) \quad \text{(by Lemma 14)} \\ &= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \sum_{L \in G_p} \sum_{\substack{K \in G \\ \theta_p(K) = L}} R_{\theta_p(K)}(n/p^2, d/p^2) \end{aligned}$$

$$= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \sum_{L \in G_p} \sum_{\substack{K \in G \\ \theta_p(K)=L}} R_L(n/p^2, d/p^2)$$

$$= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \sum_{L \in G_p} R_L(n/p^2, d/p^2) \sum_{\substack{K \in G \\ \theta_p(K)=L}} 1$$

$$= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \sum_{L \in G_p} R_L(n/p^2, d/p^2) N_G(L)$$

$$= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \cdot \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} \sum_{L \in G_p} R_L(n/p^2, d/p^2)$$

$$= \frac{\log \varepsilon(d)}{\log \varepsilon(d/p^2)} \cdot \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} R_{G_p}(n/p^2, d/p^2),$$

as asserted. ■

We now give our second reduction formula.

PROPOSITION 2. *For $G \in G(d)$, $d > 0$, we have*

$$R_G(n, d) = \frac{1}{2^{t(d)-t(d/M^2)}} \cdot \frac{\log \varepsilon(d)}{\log \varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} R_{G_M}(n/M^2, d/M^2).$$

Proof. This follows from Lemma 15 by applying it to all the primes dividing the integer M. ■

We now set

$$N(n, d) = \sum_{K \in H(d)} R_K(n, d). \tag{35}$$

For $d > 0$ Dirichlet (see for example [12: Theorem 4.1, p. 307]) has shown that

$$N(n, d) = \sum_{\nu \mid n} \left(\frac{d}{\nu}\right) \quad \text{if } (n, d) = 1.$$

Following the proof of Theorem 8.3 in [13] and using Dirichlet's result, we obtain

PROPOSITION 3. *Let $d > 0$. If $(n, d) = 1$ and $G \in G(d)$ then*

$$R_G(n, d) = \frac{1}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1).$$

We are now ready to prove Theorem 1.

Theorem 1. *Let* $G \in G(d)$, $d > 0$. *If* $\mathrm{Null}(n,d) = \emptyset$ *then*

$$R_G(n,d) = \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} \cdot \frac{1}{2^{t(d)+1}} \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1).$$

If $\mathrm{Null}(n,d) \neq \emptyset$ *then* $R_G(n,d) = 0$.

Proof. Suppose $\mathrm{Null}(n,d) = \emptyset$. By Propositions 1 and 2, we have

$$\begin{aligned} R_G(n,d) &= \frac{1}{2^{t(d)-t(d/M^2)}} \cdot \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} R_{G_M}(n/M^2, d/M^2) \\ &= \frac{1}{2^{t(d)-t(d/M^2)}} \cdot \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} R_{G_{MQ}}(n/M^2 Q, d/M^2) \end{aligned}$$

as $U(n/M^2, d/M^2) = Q$ by (33). By Lemma 6(a) we have

$$\left(\frac{n}{M^2 Q}, \frac{d}{M^2}\right) = 1 \quad \text{and} \quad \left(\frac{n}{M^2}, \frac{f}{M}\right) = 1,$$

so that, by Proposition 3, Lemma 8 and Lemma 4(b), we have

$$\begin{aligned} R_G(n,d) &= \frac{1}{2^{t(d)-t(d/M^2)}} \cdot \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} \cdot \frac{1}{2^{t(d/M^2)+1}} \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G_{MQ}) S(n/M^2 Q, d_1, d/M^2 d_1) \\ &= \frac{1}{2^{t(d)+1}} \cdot \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G_M) S(n/M^2, d_1, d/M^2 d_1) \\ &= \frac{1}{2^{t(d)+1}} \cdot \frac{\log\varepsilon(d)}{\log\varepsilon(d/M^2)} \cdot \frac{h(d)}{h(d/M^2)} \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1). \end{aligned}$$

The second assertion of Theorem 1 follows from Lemma 5. ■

3. The restricted Epstein zeta function $Z_Q(s)$. Proof of Theorem 2. Let a, b and c be integers with $a > 0$, $\gcd(a,b,c) = 1$ and $b^2 - 4ac = d$, where d is a positive nonsquare discriminant. We set $Q(x,y) = ax^2 + bxy + cy^2$ so that Q is an indefinite, primitive, integral, binary quadratic form of discriminant d. Let $\varepsilon = \varepsilon(d)$ be given by (3). For $s > 1$, we define the *restricted Epstein zeta function* $Z_Q(s)$ by

$$Z_Q(s) = \sum_{\substack{x,y=-\infty \\ Q(x,y)>0 \\ 2ax+(b-\sqrt{d})y>0 \\ 1\le\left|\frac{2ax+(b+\sqrt{d})y}{2ax+(b-\sqrt{d})y}\right|<\varepsilon^2}}^{\infty} \frac{1}{Q(x,y)^s}. \tag{36}$$

We begin by showing that the series in (36) defining $Z_Q(s)$ converges for $s > 1$. To do this, we examine the three parts of the series (36) corresponding to $y = 0$, $y > 0$ and $y < 0$, and show that each converges for $s > 1$.

The part corresponding to $y = 0$ is clearly

$$\sum_{x=1}^{\infty} \frac{1}{Q(x,0)^s} = \sum_{x=1}^{\infty} \frac{1}{(ax^2)^s} = a^{-s}\zeta(2s) \tag{37}$$

for $s > 1/2$.

For $y > 0$ we show that the conditions in the definition of $Z_Q(s)$ are satisfied if and only if $2ax > \lambda y$, where

$$\lambda = -b + \sqrt{d} + 2\sqrt{d}/(\varepsilon^2 - 1). \tag{38}$$

Set

$$E = 2ax + (b + \sqrt{d})y, \qquad E' = 2ax + (b - \sqrt{d})y.$$

The summation conditions are

$$EE' > 0, \qquad E' > 0, \qquad 1 \le |E/E'| < \varepsilon^2,$$

which are equivalent to

$$E > 0, \qquad E' > 0, \qquad E' \le E < \varepsilon^2 E'.$$

For $y > 0$ we have $E > E'$ so these conditions are equivalent to

$$E' > 0, \qquad E < \varepsilon^2 E'.$$

The second of these inequalities is equivalent to (as $\varepsilon > 1$)

$$2ax > \left(-b + \sqrt{d} + \frac{2\sqrt{d}}{\varepsilon^2 - 1}\right)y.$$

Moreover if this inequality holds then $2ax > (-b + \sqrt{d})y$ so that $E' > 0$. Hence the part corresponding to $y > 0$ is

$$\sum_{y=1}^{\infty} \sum_{x>\lambda_1 y} \frac{1}{Q(x,y)^s}, \tag{39}$$

where

$$\lambda_1 = \lambda/(2a). \tag{40}$$

If $y<0$, a short calculation similar to the above shows that the conditions in the definition of $Z_Q(s)$ are never satisfied. Thus we must examine the convergence of

$$\sum_{y=1}^{\infty}\sum_{x>\lambda_1 y}\frac{1}{Q(x,y)^s}=\sum_{y=1}^{\infty}y^{-2s}\sum_{x>\lambda_1 y}Q(xy^{-1},1)^{-s}. \tag{41}$$

To evaluate the inner sum in (41), we apply the Euler–Maclaurin summation formula. For $s>1/2$, $y>0$, we obtain

$$\begin{aligned}\sum_{x>\lambda_1 y}Q(xy^{-1},1)^{-s}&=P(\lambda_1 y)Q(\lambda_1,1)^{-s}+\int_{\lambda_1 y}^{\infty}Q(xy^{-1},1)^{-s}\,dx\\&\quad+\int_{\lambda_1 y}^{\infty}(-s)Q(xy^{-1},1)^{-s-1}(2axy^{-2}+by^{-1})P(x)\,dx\\&=y\int_{\lambda_1}^{\infty}Q(t,1)^{-s}\,dt-s\int_{\lambda_1}^{\infty}Q(t,1)^{-s-1}(2at+b)P(ty)\,dt\\&\quad+P(\lambda_1 y)Q(\lambda_1,1)^{-s},\end{aligned} \tag{42}$$

where $P(x)=x-[x]-1/2$. Thus, for $s>1$, we have

$$\begin{aligned}&\sum_{y=1}^{\infty}y^{-2s}\sum_{x>\lambda_1 y}Q(xy^{-1},1)^{-s}\\&\quad=\zeta(2s-1)\int_{\lambda_1}^{\infty}Q(t,1)^{-s}\,dt+Q(\lambda_1,1)^{-s}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^{2s}}\\&\qquad-s\sum_{y=1}^{\infty}y^{-2s}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^{s+1}}\,dt.\end{aligned} \tag{43}$$

This shows that

$$\sum_{y=1}^{\infty}y^{-2s}\sum_{x>\lambda_1 y}Q(xy^{-1},1)^{-s},$$

and thus the original series for $Z_Q(s)$ converges for $s>1$. Putting together (36), (37) and (43), we obtain

Lemma 16. *For $s>1$, we have*

$$\begin{aligned}Z_Q(s)&=a^{-s}\zeta(2s)+\zeta(2s-1)\int_{\lambda_1}^{\infty}Q(t,1)^{-s}\,dt+Q(\lambda_1,1)^{-s}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^{2s}}\\&\quad-s\sum_{y=1}^{\infty}y^{-2s}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^{s+1}}\,dt.\end{aligned} \tag{44}$$

We are now ready to prove Theorem 2.

THEOREM 2. *Let d be a positive nonsquare discriminant. Let $Q=(a,b,c)$ be a primitive, integral, binary quadratic form of discriminant d with $a>0$. Let $\varepsilon=\frac{1}{2}(x_0+y_0\sqrt{d})$ be as defined in* (3). *Set*

$$\alpha=(x_0-by_0)/2, \qquad g=ay_0.$$

Then $\alpha\in\mathbb{Z}$ and $(\alpha,g)=1$. Define $\alpha'\in\mathbb{Z}$ by

$$\alpha\alpha'\equiv 1 \pmod{g}, \qquad 0\le\alpha'<g.$$

For $l=1,\dots,[(g-1)/2]$ define $l^*\in\mathbb{Z}$ by

$$l\alpha\equiv l^* \pmod{g}, \qquad 0\le l^*<g.$$

For $l=1,\dots,[(g-1)/2]$ and $0\le t\le 1$ set

$$F(\alpha,l,t,g)=\frac{(t-\cos(2\pi l\alpha/g))\log(1-2t^{\varepsilon}\cos(2\pi l/g)+t^{2\varepsilon})}{t^2-2t\cos(2\pi l\alpha/g)+1} - \frac{2\sin(2\pi l\alpha/g)\tan^{-1}\left(\dfrac{t^{\varepsilon}\sin(2\pi l/g)}{1-t^{\varepsilon}\cos(2\pi l/g)}\right)}{t^2-2t\cos(2\pi l\alpha/g)+1}.$$

Then

$$Z_Q(s)=\frac{\log\varepsilon}{\sqrt{d}}\cdot\frac{1}{s-1}+C_Q+O(s-1) \qquad \text{as } s\to 1^+,$$

where

$$C_Q=V(d)+\frac{\pi^2}{6a}+\frac{\log\varepsilon\,\log a}{\sqrt{d}}-\frac{1}{\sqrt{d}}W_Q,$$

$$V(d)=\frac{2\gamma\log\varepsilon}{\sqrt{d}}+\frac{\log\varepsilon\,\log(\varepsilon y_0^2)}{\sqrt{d}}-\frac{1}{2\sqrt{d}}\int_0^{\infty}\left(\frac{\log(u+\varepsilon^2)}{u+1}-\frac{\log(u+1)}{u+\varepsilon^2}\right)du + \frac{1}{\sqrt{d}}\int_0^1\left(\frac{1}{t\log t}-\frac{1}{t-1}\right)\log\left(\frac{1-t^{\varepsilon}}{1-t^{\varepsilon'}}\right)dt,$$

and

$$W_Q=\int_0^1\sum_{l=1}^{[(g-1)/2]}(F(\alpha,l,t,g)+F(\alpha',l,t,g))\,dt - 2\sum_{l=1}^{[(g-1)/2]}\left(\log\left(2\sin\frac{\pi l}{g}\right)\log\left(2\left|\sin\frac{\pi l\alpha}{g}\right|\right)-\left(\frac{\pi}{2}-\frac{\pi l}{g}\right)\left(\frac{\pi}{2}-\frac{\pi l^*}{g}\right)\right) + \left(2\int_0^1\frac{\log(1+t^{\varepsilon})}{1+t}\,dt-\log^2 2\right)\left(\frac{1+(-1)^g}{2}\right).$$

Proof. All the series and integrals appearing in Lemma 16 except the series for $\zeta(2s-1)$ regarded as functions of the complex variable s converge uniformly on compact subsets of the region $\mathrm{Re}(s) > 1/2$, and so are analytic in this region. As $s \to 1^+$, we have

$$\tag{45} \int_{\lambda_1}^{\infty} Q(t,1)^{-s}\,dt = \int_{\lambda_1}^{\infty} \frac{1}{Q(t,1)}\,dt - \left(\int_{\lambda_1}^{\infty} \frac{\log Q(t,1)}{Q(t,1)}\,dt\right)(s-1) + O((s-1)^2).$$

We have $Q(t,1) = a(t+t_1)(t+t_2)$, where

$$\tag{46} t_1 = \frac{b+\sqrt{d}}{2a}, \qquad t_2 = \frac{b-\sqrt{d}}{2a}.$$

We note that by (38) and (40), $t+t_1$ and $t+t_2$ are positive for $t \geq \lambda_1$. Using these facts, it is easily shown that

$$\tag{47} \int_{\lambda_1}^{\infty} \frac{1}{Q(t,1)}\,dt = \frac{2\log\varepsilon}{\sqrt{d}}.$$

Also

$$\begin{aligned}
\int_{\lambda_1}^{\infty} \frac{\log Q(t,1)}{Q(t,1)}\,dt
&= \frac{1}{a(t_1-t_2)} \int_{\lambda_1}^{\infty} \log(a(t+t_1)(t+t_2))\left(\frac{1}{t+t_2} - \frac{1}{t+t_1}\right) dt \\
&= \frac{1}{\sqrt{d}} \int_{\lambda_1}^{\infty} \log a \left(\frac{1}{t+t_2} - \frac{1}{t+t_1}\right) dt \\
&\quad + \frac{1}{\sqrt{d}} \int_{\lambda_1}^{\infty} \log((t+t_1)(t+t_2))\left(\frac{1}{t+t_2} - \frac{1}{t+t_1}\right) dt \\
&= \frac{1}{\sqrt{d}} \int_{0}^{\infty} \log((t+\lambda_1+t_1)(t+\lambda_1+t_2))\left(\frac{1}{t+\lambda_1+t_2} - \frac{1}{t+\lambda_1+t_1}\right) dt \\
&\quad + \frac{2\log a\,\log\varepsilon}{\sqrt{d}}.
\end{aligned}$$

Let

$$\tag{48} \lambda_0 = \frac{\sqrt{d}}{a(\varepsilon^2-1)},$$

so that by (38), (40) and (46), we have

$$\lambda_1 + t_1 = \varepsilon^2\lambda_0, \qquad \lambda_1 + t_2 = \lambda_0. \tag{49}$$

Hence

$$\begin{aligned}
\int_0^\infty \log((t+\lambda_1+t_1)(t+\lambda_1+t_2))\left(\frac{1}{t+\lambda_1+t_2} - \frac{1}{t+\lambda_1+t_1}\right) dt \\
= \int_0^\infty \log((t+\varepsilon^2\lambda_0)(t+\lambda_0))\left(\frac{1}{t+\lambda_0} - \frac{1}{t+\varepsilon^2\lambda_0}\right) dt \\
= \int_0^\infty \left(\frac{\log(t+\lambda_0)}{t+\lambda_0} - \frac{\log(t+\varepsilon^2\lambda_0)}{t+\varepsilon^2\lambda_0}\right) dt \\
+ \int_0^\infty \left(\frac{\log(t+\varepsilon^2\lambda_0)}{t+\lambda_0} - \frac{\log(t+\lambda_0)}{t+\varepsilon^2\lambda_0}\right) dt \\
= 2\log\varepsilon\ \log(\varepsilon\lambda_0) + \int_0^\infty \left(\frac{\log(t+\varepsilon^2\lambda_0)}{t+\lambda_0} - \frac{\log(t+\lambda_0)}{t+\varepsilon^2\lambda_0}\right) dt \\
= 2\log\varepsilon\ \log(\varepsilon\lambda_0) + 2\log\varepsilon\ \log\lambda_0 \\
+ \int_0^\infty \left(\frac{\log(u+\varepsilon^2)}{u+1} - \frac{\log(u+1)}{u+\varepsilon^2}\right) du,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{\lambda_1}^\infty \frac{\log Q(t,1)}{Q(t,1)}\, dt \\
= \frac{2\log\varepsilon\ \log(a\varepsilon\lambda_0^2)}{\sqrt{d}} + \frac{1}{\sqrt{d}}\int_0^\infty \left(\frac{\log(u+\varepsilon^2)}{u+1} - \frac{\log(u+1)}{u+\varepsilon^2}\right) du.
\end{aligned} \tag{50}$$

Using (45), (47) and (50), together with

$$a^{-s}\zeta(2s) = \frac{\pi^2}{6a} + O(s-1),$$

$$\zeta(2s-1) = \frac{1/2}{s-1} + \gamma + O(s-1),$$

$$Q(\lambda_1,1)^{-s}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^{2s}} = Q(\lambda_1,1)^{-1}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^2} + O(s-1),$$

$$s\sum_{y=1}^{\infty} y^{-2s}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^{s+1}}\,dt=\sum_{y=1}^{\infty} y^{-2}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^{2}}\,dt+O(s-1),$$

in (44), where γ denotes Euler's constant

$$\gamma=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i}-\log n\right)=0.5772156649\ldots,$$

we obtain

$$Z_Q(s)=\frac{\log\varepsilon}{\sqrt{d}}\cdot\frac{1}{s-1}+C_Q+O(s-1)\quad\text{as } s\to 1^{+}, \tag{51}$$

where

$$\begin{aligned}C_Q=&\frac{\pi^2}{6a}+\frac{2\gamma\log\varepsilon}{\sqrt{d}}-\frac{1}{2\sqrt{d}}\int_{0}^{\infty}\left(\frac{\log(u+\varepsilon^2)}{u+1}-\frac{\log(u+1)}{u+\varepsilon^2}\right)du\\&-\frac{\log\varepsilon\,\log(a\varepsilon\lambda_0^2)}{\sqrt{d}}+Q(\lambda_1,1)^{-1}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^2}\\&-\sum_{y=1}^{\infty}y^{-2}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^2}\,dt.\end{aligned} \tag{52}$$

Set

$$K(d)=\frac{2\gamma\log\varepsilon}{\sqrt{d}}-\frac{1}{2\sqrt{d}}\int_{0}^{\infty}\left(\frac{\log(u+\varepsilon^2)}{u+1}-\frac{\log(u+1)}{u+\varepsilon^2}\right)du, \tag{53}$$

so that

$$\begin{aligned}C_Q=&K(d)+\frac{\pi^2}{6a}-\frac{\log\varepsilon\,\log(a\varepsilon\lambda_0^2)}{\sqrt{d}}+\frac{1}{Q(\lambda_1,1)}\sum_{y=1}^{\infty}\frac{P(\lambda_1 y)}{y^2}\\&-\sum_{y=1}^{\infty}y^{-2}\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^2}\,dt.\end{aligned} \tag{54}$$

We emphasize that $K(d)$ depends only on d and not on the form (a,b,c).

Throughout the rest of this section, we focus on transforming C_Q into the form stated in Theorem 2. By (42) and (47), we have for $y>0$,

$$\begin{aligned}\sum_{x>\lambda_1 y}Q(xy^{-1},1)^{-1}&=y\int_{\lambda_1}^{\infty}\frac{1}{Q(t,1)}\,dt-\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^2}\,dt+\frac{P(\lambda_1 y)}{Q(\lambda_1,1)}\\&=\frac{2y\log\varepsilon}{\sqrt{d}}-\int_{\lambda_1}^{\infty}\frac{(2at+b)P(ty)}{Q(t,1)^2}\,dt+\frac{P(\lambda_1 y)}{Q(\lambda_1,1)}.\end{aligned}$$

Hence

$$\sum_{y=1}^{\infty} y^{-2}\Big(\sum_{x>\lambda_1 y} Q(xy^{-1},1)^{-1} - \frac{2y\log\varepsilon}{\sqrt{d}}\Big) = \sum_{y=1}^{\infty} y^{-2}\Big(\frac{P(\lambda_1 y)}{Q(\lambda_1,1)} - \int_{\lambda_1}^{\infty} \frac{(2at+b)P(ty)}{Q(t,1)^2}\,dt\Big).$$

Using this in (54) gives

$$C_Q = K(d) + \frac{\pi^2}{6a} - \frac{\log\varepsilon\ \log(a\varepsilon\lambda_0^2)}{\sqrt{d}} + \sum_{y=1}^{\infty} y^{-2}\Big(\sum_{x>\lambda_1 y} Q(xy^{-1},1)^{-1} - \frac{2y\log\varepsilon}{\sqrt{d}}\Big). \tag{55}$$

By (46), we have, for $y>0$,

$$\begin{aligned}\sum_{x>\lambda_1 y} Q(xy^{-1},1)^{-1} &= \sum_{x>\lambda_1 y} \frac{1}{a(xy^{-1}+t_1)(xy^{-1}+t_2)} \\ &= \frac{y}{\sqrt{d}} \sum_{x=1+[\lambda_1 y]}^{\infty} \Big(\frac{1}{x+t_2 y} - \frac{1}{x+t_1 y}\Big) \\ &= \frac{y}{\sqrt{d}} \sum_{m=0}^{\infty} \Big(\frac{1}{m+1+[\lambda_1 y]+t_2 y} - \frac{1}{m+1+[\lambda_1 y]+t_1 y}\Big).\end{aligned} \tag{56}$$

Using (49), we note that $1+[\lambda_1 y]+t_2 y > (\lambda_1+t_2)y = \lambda_0 y > 0$ and $1+[\lambda_1 y]+t_1 y > (\lambda_1+t_1)y = \lambda_0\varepsilon^2 y > 0$. We recall the formula (see for example [10: formula 8.362, p. 952]), which is valid for $x>0$,

$$-\psi(x) = \frac{1}{x} + \gamma + \sum_{m=1}^{\infty}\Big(\frac{1}{x+m} - \frac{1}{m}\Big),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Hence, for $x_1>0, x_2>0$, we have

$$\psi(x_1) - \psi(x_2) = \sum_{m=0}^{\infty}\Big(\frac{1}{m+x_2} - \frac{1}{m+x_1}\Big).$$

Using this in (56), we obtain, for $y>0$,

$$\sum_{x>\lambda_1 y} Q(xy^{-1},1)^{-1} = \frac{y}{\sqrt{d}}(\psi(1+[\lambda_1 y]+t_1 y) - \psi(1+[\lambda_1 y]+t_2 y)). \tag{57}$$

Hence, by (55), we have

$$C_Q = K(d) + \frac{\pi^2}{6a} - \frac{\log\varepsilon\ \log(a\varepsilon\lambda_0^2)}{\sqrt{d}} + \frac{1}{\sqrt{d}}\sum_{y=1}^{\infty}\frac{1}{y}(\psi(1+[\lambda_1 y]+t_1 y) - \psi(1+[\lambda_1 y]+t_2 y) - \log\varepsilon^2). \tag{58}$$

As in the proof of Lemma 13, we set

(59) $$\varepsilon' = (x_0 - y_0\sqrt{d})/2,$$

so that

(60) $$\varepsilon\varepsilon' = 1$$

as $x_0^2 - dy_0^2 = 4$. Hence $\varepsilon'(\varepsilon^2 - 1) = \varepsilon - \varepsilon' = y_0\sqrt{d}$, so that by (48),

(61) $$\lambda_0 = \frac{\sqrt{d}}{a(\varepsilon^2 - 1)} = \frac{\varepsilon'}{ay_0},$$

and by (38),

$$\lambda = -b + \sqrt{d} + \frac{2\sqrt{d}}{\varepsilon^2 - 1} = -b + \frac{\varepsilon - \varepsilon'}{y_0} + \frac{2\varepsilon'}{y_0} = -b + \frac{\varepsilon + \varepsilon'}{y_0}.$$

Hence

(62) $$\lambda = -b + x_0/y_0$$

and by (40),

(63) $$\lambda_1 = \lambda/(2a) = \alpha/g,$$

where

(64) $$g = g(a, d) = ay_0, \qquad \alpha = \alpha(b, d) = (x_0 - by_0)/2.$$

Since $b^2 - 4ac = d$ and $x_0^2 - dy_0^2 = 4$, we have

$$x_0^2 - b^2y_0^2 = 4 - 4acy_0^2 \equiv 0 \pmod 4,$$

so that $x_0 \equiv by_0 \pmod 2$. Hence α is an integer. In fact

(65) $$(\alpha, g) = 1,$$

since

$$\alpha\left(\frac{x_0 + by_0}{2}\right) + gcy_0 = 1.$$

We have, by (40), (46), (62), (3) and (64),

(66) $$\lambda_1 + t_1 = \frac{\lambda + b + \sqrt{d}}{2a} = \frac{x_0/y_0 + \sqrt{d}}{2a} = \frac{\varepsilon}{g},$$

and similarly

(67) $$\lambda_1 + t_2 = \varepsilon'/g.$$

For any positive integer y, we let

(68) $$r_y = g(\lambda_1 y - [\lambda_1 y]).$$

We note that r_y is an integer since $r_y = \alpha y - g[\lambda_1 y]$ by (63). Also

(69) $$0 \le r_y < g.$$

For $x_1 > 0$ and $x_2 > 0$ we have

$$\log x_1 - \log x_2 = \int_{x_2}^{x_1} \frac{1}{t}\,dt = \int_{x_2}^{x_1}\int_0^{\infty} e^{-tu}\,du\,dt$$

so that

$$\log x_1 - \log x_2 = \int_0^{\infty}\int_{x_2}^{x_1} e^{-tu}\,dt\,du = \int_0^{\infty} \frac{e^{-x_2 u} - e^{-x_1 u}}{u}\,du.$$

On using the substitution $t = e^{-u}$, we obtain

$$\log x_1 - \log x_2 = \int_0^1 \frac{t^{x_1} - t^{x_2}}{t\log t}\,dt. \tag{70}$$

Equation 3.311(6) in [10] gives

$$\psi(x) + \gamma = \int_0^{\infty} \frac{e^{-u} - e^{-xu}}{1 - e^{-u}}\,du = \int_0^1 \frac{1 - t^{x-1}}{1 - t}\,dt, \qquad x > 0.$$

Choosing $x = 1 + x_1 > 0$ and $x = 1 + x_2 > 0$, and subtracting, we obtain

$$\psi(1 + x_1) - \psi(1 + x_2) = \int_0^1 \frac{t^{x_2} - t^{x_1}}{1 - t}\,dt. \tag{71}$$

Appealing to (60) and (66)–(71), we obtain

$$\begin{aligned}
&\sum_{y=1}^{\infty} \frac{1}{y}(\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y) - \log\varepsilon^2)\\
&\quad= \sum_{y=1}^{\infty} \frac{1}{y}(\psi(1 + (\lambda_1 + t_1)y - (\lambda_1 y - [\lambda_1 y]))\\
&\qquad - \psi(1 + (\lambda_1 + t_2)y - (\lambda_1 y - [\lambda_1 y])) - \log\varepsilon^2)\\
&\quad= \sum_{y=1}^{\infty} \frac{1}{y}\left(\psi\left(1 + \frac{\varepsilon y}{g} - \frac{r_y}{g}\right) - \psi\left(1 + \frac{\varepsilon' y}{g} - \frac{r_y}{g}\right) - (\log(\varepsilon y) - \log(\varepsilon' y))\right)\\
&\quad= \sum_{y=1}^{\infty} \frac{1}{y}\left(\int_0^1 \frac{t^{(\varepsilon' y - r_y)/g} - t^{(\varepsilon y - r_y)/g}}{1 - t}\,dt - \int_0^1 \frac{t^{\varepsilon y} - t^{\varepsilon' y}}{t\log t}\,dt\right)\\
&\quad= \sum_{y=1}^{\infty} \frac{1}{y}\left(g\int_0^1 \frac{t^{\varepsilon' y} - t^{\varepsilon y}}{1 - t^g}\,t^{g-1-r_y}\,dt - \int_0^1 \frac{t^{\varepsilon y} - t^{\varepsilon' y}}{t\log t}\,dt\right)\\
&\quad= \int_0^1 \sum_{y=1}^{\infty} \frac{t^{\varepsilon y} - t^{\varepsilon' y}}{ty}\left(-\frac{1}{\log t} + \frac{g t^{g - r_y}}{t^g - 1}\right)dt,
\end{aligned}$$

so that

$$\sum_{y=1}^{\infty} \frac{1}{y}\left(\psi(1+[\lambda_1 y]+t_1 y)-\psi(1+[\lambda_1 y]+t_2 y)-\log\varepsilon^2\right) = \int_0^1 \left(\frac{1}{t\log t}\log\left(\frac{1-t^{\varepsilon}}{1-t^{\varepsilon'}}\right)+g\sum_{y=1}^{\infty}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y(t^g-1)}\,t^{g-r_y-1}\right)dt. \tag{72}$$

Let $0<t<1$. We have

$$\begin{aligned}\sum_{y=1}^{\infty}\frac{g}{y}\cdot\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{t^g-1}\,t^{g-r_y-1} &= \sum_{k=0}^{g-1}\sum_{r_y=k}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\cdot\frac{gt^{g-1-k}}{t^g-1}\\ &= \sum_{k=0}^{g-1}\sum_{\alpha y\equiv k\,(\mathrm{mod}\,g)}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\cdot\frac{gt^{g-k-1}}{t^g-1}.\end{aligned} \tag{73}$$

Let

$$\theta = e^{2\pi i/g}. \tag{74}$$

Then

$$\frac{gt^{g-k-1}}{t^g-1} = \frac{gt^{g-k-1}}{(t-\theta)\cdots(t-\theta^g)} = \sum_{l=1}^{g}\frac{A_{l,k}}{t-\theta^l}, \tag{75}$$

where $A_{l,k}=g\theta^{-l(k+1)}/\prod_{j\neq l}(\theta^l-\theta^j)$. But, for $1\le l\le g$, we have

$$\prod_{j\neq l}(\theta^l-\theta^j)=\theta^{l(g-1)}\prod_{j\neq l}(1-\theta^{j-l})=\theta^{-l}\prod_{i=1}^{g-1}(1-\theta^i)=g\theta^{-l},$$

so that

$$A_{l,k}=\theta^{-lk}. \tag{76}$$

Thus, by (73), (75) and (76), we have

$$\begin{aligned}\sum_{y=1}^{\infty}\frac{g}{y}\cdot\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{t^g-1}\,t^{g-r_y-1} &= \sum_{k=0}^{g-1}\sum_{\alpha y\equiv k\,(\mathrm{mod}\,g)}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\sum_{l=1}^{g}\frac{\theta^{-lk}}{t-\theta^l}\\ &= \sum_{l=1}^{g}\frac{1}{t-\theta^l}\sum_{k=0}^{g-1}\sum_{\alpha y\equiv k\,(\mathrm{mod}\,g)}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\,\theta^{-lk}\\ &= \sum_{l=1}^{g}\frac{1}{t-\theta^l}\sum_{k=0}^{g-1}\sum_{\alpha y\equiv k\,(\mathrm{mod}\,g)}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\,\theta^{-l\alpha y}\\ &= \sum_{l=1}^{g}\frac{1}{t-\theta^l}\sum_{y=1}^{\infty}\frac{t^{\varepsilon y}-t^{\varepsilon' y}}{y}\,\theta^{-l\alpha y}\end{aligned}$$

$$= -\sum_{l=1}^{g} \frac{1}{t-\theta^l}(\log(1-\theta^{-l\alpha}t^{\varepsilon}) - \log(1-\theta^{-l\alpha}t^{\varepsilon'})),$$

where the principal values of the logarithms are taken. Using this in (72) gives

$$(77)\qquad \sum_{y=1}^{\infty} \frac{1}{y}\,(\psi(1+[\lambda_1 y]+t_1 y) - \psi(1+[\lambda_1 y]+t_2 y) - \log\varepsilon^2)$$
$$= \int_0^1 \bigg(\frac{1}{t\log t}\log\bigg(\frac{1-t^{\varepsilon}}{1-t^{\varepsilon'}}\bigg) - \sum_{l=1}^{g} \frac{1}{t-\theta^l}\,(\log(1-\theta^{-l\alpha}t^{\varepsilon}) - \log(1-\theta^{-l\alpha}t^{\varepsilon'}))\bigg)\,dt$$
$$= \int_0^1 \bigg(\frac{1}{t\log t} - \frac{1}{t-1}\bigg)\log\bigg(\frac{1-t^{\varepsilon}}{1-t^{\varepsilon'}}\bigg)\,dt - \int_0^1\sum_{l=1}^{g-1} \frac{1}{t-\theta^l}\,(\log(1-\theta^{-l\alpha}t^{\varepsilon}) - \log(1-\theta^{-l\alpha}t^{\varepsilon'}))\,dt.$$

For $1 \le l \le g-1$, we have $\theta^l \ne 1$, $\theta^{-l\alpha} \ne 1$ (by (65)), and

$$(78)\qquad \int_0^1 \frac{1}{t-\theta^l}\log(1-\theta^{-l\alpha}t^{\varepsilon'})\,dt = -\theta^{-l}\int_0^1 \frac{1}{1-\theta^{-l}t}\log(1-\theta^{-l\alpha}t^{\varepsilon'})\,dt$$
$$= [\log(1-\theta^{-l\alpha}t^{\varepsilon'})\log(1-\theta^{-l}t)]_0^1 - \int_0^1 \frac{\log(1-\theta^{-l}t)}{1-\theta^{-l\alpha}t^{\varepsilon'}}\,(-\theta^{-l\alpha})\varepsilon' t^{\varepsilon'-1}\,dt$$
$$= \log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) - \int_0^1 \frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}\,dt.$$

Hence, by (78), (77), (58), (61), (60) and (53), we obtain

$$(79)\qquad C_Q = V(d) + \frac{\pi^2}{6a} + \frac{\log\varepsilon\,\log a}{\sqrt{d}} - \frac{1}{\sqrt{d}}\,W_Q,$$

where $V(d)$ is defined in the statement of Theorem 2 and

$$(80)\qquad W_Q = \sum_{l=1}^{g-1}\bigg(\int_0^1 \frac{\log(1-\theta^{-l\alpha}t^{\varepsilon})}{t-\theta^l}\,dt + \int_0^1 \frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}\,dt\bigg) - \sum_{l=1}^{g-1}\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}).$$

We emphasize that $V(d)$ depends only on d and not on the form (a,b,c).

Since $(\alpha,g)=1$, we may choose an integer α' such that

$$\alpha\alpha' \equiv 1 \pmod{g}. \tag{81}$$

Changing the variable from l to $l\alpha'$ in the first sum in (80), we obtain

$$\begin{aligned} W_Q &= \sum_{l=1}^{g-1}\int_0^1 \log(1-\theta^{-l}t^{\varepsilon})\left(\frac{1}{t-\theta^{l\alpha'}}+\frac{1}{t-\theta^{l\alpha}}\right)dt \\ &\quad - \sum_{l=1}^{g-1}\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) \\ &= S_1(\alpha)+S_1(\alpha')-\sum_{l=1}^{g-1}\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}), \end{aligned} \tag{82}$$

where

$$S_1(\alpha)=\sum_{l=1}^{g-1}\int_0^1 \frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}\,dt. \tag{83}$$

We set

$$F(\alpha,l,t,g)=\frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}+\frac{\log(1-\theta^{l}t^{\varepsilon})}{t-\theta^{-l\alpha}}. \tag{84}$$

We first consider the case when g is odd. Let $g=2m+1$ where $m\ge 1$. We note that $W_Q=0$ if $g=1$. Then

$$\begin{aligned} S_1(\alpha) &= \sum_{l=1}^{2m}\int_0^1 \frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}\,dt \\ &= \int_0^1\left(\sum_{l=1}^{m}\frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}+\sum_{l=m+1}^{2m}\frac{\log(1-\theta^{2m+1-l}t^{\varepsilon})}{t-\theta^{-(2m+1-l)\alpha}}\right)dt \\ &= \int_0^1\sum_{l=1}^{m}\left(\frac{\log(1-\theta^{-l}t^{\varepsilon})}{t-\theta^{l\alpha}}+\frac{\log(1-\theta^{l}t^{\varepsilon})}{t-\theta^{-l\alpha}}\right)dt \\ &= \int_0^1\sum_{l=1}^{m}F(\alpha,l,t,g)\,dt. \end{aligned}$$

Similarly we have

$$\begin{aligned} &\sum_{l=1}^{g-1}\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) \\ &\qquad = \sum_{l=1}^{m}(\log(1-\theta^{-l\alpha})\log(1-\theta^{-l})+\log(1-\theta^{l\alpha})\log(1-\theta^{l})). \end{aligned}$$

Hence, by (82), we have

$$W_Q = \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g))\,dt - \sum_{l=1}^{[(g-1)/2]} (\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) + \log(1-\theta^{l\alpha})\log(1-\theta^{l})).$$

Similarly, for g even, we obtain

$$W_Q = \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g))\,dt + 2\int_0^1 \frac{\log(1+t^{\varepsilon})}{1+t}\,dt - \sum_{l=1}^{[(g-1)/2]} (\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) + \log(1-\theta^{l\alpha})\log(1-\theta^{l})) - \log^2 2.$$

Thus, for all g, we have

$$(85)\quad W_Q = \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g))\,dt - \sum_{l=1}^{[(g-1)/2]} (\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) + \log(1-\theta^{l\alpha})\log(1-\theta^{l})) + \left(2\int_0^1 \frac{\log(1+t^{\varepsilon})}{1+t}\,dt - \log^2 2\right)\frac{1+(-1)^g}{2}.$$

Explicitly calculating the logarithms occurring in (84), we obtain, after some simplification,

$$(86)\quad F(\alpha, l, t, g) = \frac{(t-\cos(2\pi l\alpha/g))\log(1-2t^{\varepsilon}\cos(2\pi l/g)+t^{2\varepsilon})}{t^2-2t\cos(2\pi l\alpha/g)+1} - \frac{2\sin(2\pi l\alpha/g)\tan^{-1}\left(\dfrac{t^{\varepsilon}\sin(2\pi l/g)}{1-t^{\varepsilon}\cos(2\pi l/g)}\right)}{t^2-2t\cos(2\pi l\alpha/g)+1},$$

for $1 \le l \le [(g-1)/2]$, $0 \le t \le 1$. Similarly, for $1 \le l \le [(g-1)/2]$, we obtain

$$\log(1-\theta^{-l\alpha})\log(1-\theta^{-l}) + \log(1-\theta^{l\alpha})\log(1-\theta^{l}) = 2\left(\log\left(2\sin\frac{\pi l}{g}\right)\log\left(2\left|\sin\frac{\pi l\alpha}{g}\right|\right) - \left(\frac{\pi}{2}-\frac{\pi l}{g}\right)\tan^{-1}\left(\cot\frac{\pi l\alpha}{g}\right)\right).$$

Let

$$(87)\qquad l\alpha \equiv l^* \pmod g,$$

where $0 \le l^* < g$. Then

$$\tan^{-1}\left(\cot\frac{\pi l\alpha}{g}\right) = \frac{\pi}{2} - \frac{\pi l^*}{g}.$$

Thus our final formula for W_Q is

$$W_Q = \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha,l,t,g) + F(\alpha',l,t,g))\,dt \tag{88}$$
$$- 2\sum_{l=1}^{[(g-1)/2]} \left(\log\left(2\sin\frac{\pi l}{g}\right)\log\left(2\left|\sin\frac{\pi l\alpha}{g}\right|\right)\right.$$
$$\left. - \left(\frac{\pi}{2} - \frac{\pi l}{g}\right)\left(\frac{\pi}{2} - \frac{\pi l^*}{g}\right)\right)$$
$$+ \left(2\int_0^1 \frac{\log(1+t^\varepsilon)}{1+t}\,dt - \log^2 2\right)\left(\frac{1+(-1)^g}{2}\right).$$

This completes our proof of Theorem 2. ∎

4. Behaviour of $\sum_{n=1}^\infty R_G(n,d)/n^s$ near $s=1$. Proofs of Theorems 3 and 4. Let $K \in H(d)$, where d is a positive nonsquare discriminant, and let $Q=(a,b,c) \in K$ with $a>0$. For $s>1$ we have

$$Z_Q(s) = \sum_{n=1}^\infty \frac{R_Q(n,d)}{n^s} = \sum_{n=1}^\infty \frac{R_K(n,d)}{n^s}. \tag{89}$$

Thus, for $G \in G(d)$, we see that

$$\sum_{n=1}^\infty \frac{R_G(n,d)}{n^s} = \sum_{K\in G}\sum_{n=1}^\infty \frac{R_K(n,d)}{n^s} \tag{90}$$

converges for $s>1$. We now evaluate the Dirichlet series on the left hand side of (90) explicitly using the formula for $R_G(n,d)$ given in Theorem 1. We prove

THEOREM 3. *Let $G \in G(d)$. For $s>1$, we have*

$$\sum_{n=1}^\infty \frac{R_G(n,d)}{n^s}$$
$$= \frac{h(d)\log\varepsilon(d)}{2^{t(d)+1}}\sum_{m\mid f}\frac{1}{\log\varepsilon(d/m^2)h(d/m^2)}\cdot\frac{1}{m^{2s}}\sum_{d_1\in F(d/m^2)}\gamma_{d_1}(G)$$
$$\times \prod_{p\mid f/m}\left(1-\left(\frac{d_1}{p}\right)p^{-s}\right)\left(1-\left(\frac{\Delta(d/d_1)}{p}\right)p^{-s}\right)L(s,d_1)L(s,\Delta(d/d_1)),$$

where the Dirichlet L-series $L(s,d)$ is defined for $s>0$ by

$$L(s,d)=\sum_{n=1}^{\infty}\frac{\left(\frac{d}{n}\right)}{n^s}.$$

Proof. By Theorem 1, we have

$$\begin{aligned}(91)\quad &\sum_{n=1}^{\infty}\frac{R_G(n,d)}{n^s}\\ &=\sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset}}^{\infty}\frac{1}{n^s}\cdot\frac{\log\varepsilon(d)}{\log\varepsilon(d/M(n,d)^2)}\cdot\frac{h(d)}{h(d/M(n,d)^2)}\cdot\frac{1}{2^{t(d)+1}}\\ &\quad\times\sum_{d_1\in F(d/M(n,d)^2)}\gamma_{d_1}(G)S(n/M(n,d)^2,d_1,d/M(n,d)^2d_1)\\ &=\frac{h(d)\log\varepsilon(d)}{2^{t(d)+1}}\sum_{m\mid f}\frac{1}{\log\varepsilon(d/m^2)h(d/m^2)}\sum_{d_1\in F(d/m^2)}\gamma_{d_1}(G)\\ &\quad\times\sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty}\frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}.\end{aligned}$$

For $m^2\mid n$ and $m\mid f$ it is easy to check that

$$\begin{aligned}\mathrm{Null}(n,d)=\emptyset\ &\Leftrightarrow\ \mathrm{Null}(n/m^2,d/m^2)=\emptyset,\\ M(n,d)=m\ &\Leftrightarrow\ M(n/m^2,d/m^2)=1.\end{aligned}$$

Hence for $m\mid f$ we have

$$\begin{aligned}\sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty}\frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}&=\sum_{\substack{n=1\\ m^2\mid n\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty}\frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}\\ &=\sum_{\substack{n=1\\ m^2\mid n\\ \mathrm{Null}(n/m^2,d/m^2)=\emptyset\\ M(n/m^2,d/m^2)=1}}^{\infty}\frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}\\ &=\sum_{\substack{n=1\\ m^2\mid n\\ (n/m^2,f/m)=1}}^{\infty}\frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}\quad\text{(by Lemma 6)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{N=1\\(N,f/m)=1}}^{\infty} \frac{S(N,d_1,d/m^2d_1)}{(m^2N)^s}\\
&= m^{-2s} \sum_{\substack{N=1\\(N,f/m)=1}}^{\infty} \frac{1}{N^s} \sum_{\mu\nu=N} \left(\frac{d_1}{\mu}\right)\left(\frac{d/m^2d_1}{\nu}\right)\\
&= m^{-2s} \sum_{(\mu,f/m)=1} \frac{1}{\mu^s}\left(\frac{d_1}{\mu}\right) \sum_{(\nu,f/m)=1} \frac{1}{\nu^s}\left(\frac{d/m^2d_1}{\nu}\right).
\end{aligned}
$$

As $d_1 \in F(d/m^2)$, by Lemma 1(d) we have

$$
f(d_2) \mid f/m, \quad \text{where} \quad d_2 = \frac{d/m^2}{d_1}.
$$

Thus for $(\nu, f/m) = 1$ we have $(\nu, f(d_2)) = 1$ so that

$$
\begin{aligned}
\left(\frac{d/m^2d_1}{\nu}\right) &= \left(\frac{d_2}{\nu}\right) = \left(\frac{\Delta(d_2)f(d_2)^2}{\nu}\right) = \left(\frac{\Delta(d_2)}{\nu}\right)\\
&= \left(\frac{\Delta(d_2m^2)}{\nu}\right) = \left(\frac{\Delta(d/d_1)}{\nu}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty} \frac{S(n/m^2,d_1,d/m^2d_1)}{n^s}\\
&\qquad = m^{-2s} \sum_{(\mu,f/m)=1} \frac{1}{\mu^s}\left(\frac{d_1}{\mu}\right) \sum_{(\nu,f/m)=1} \frac{1}{\nu^s}\left(\frac{\Delta(d/d_1)}{\nu}\right)\\
&\qquad = m^{-2s} L(s,d_1) \prod_{p\mid f/m}\left(1-\left(\frac{d_1}{p}\right)p^{-s}\right)\\
&\qquad\quad \times L(s,\Delta(d/d_1)) \prod_{p\mid f/m}\left(1-\left(\frac{\Delta(d/d_1)}{p}\right)p^{-s}\right).
\end{aligned}
$$

The required result now follows on using (91). ∎

We next determine the behaviour of $\sum_{n=1}^{\infty} R_G(n,d)/n^s$ as $s \to 1^+$.

THEOREM 4. *Let $G \in G(d)$, $d > 0$. As $s \to 1^+$ we have*

$$
\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{h(d)\log\varepsilon(d)}{2^{t(d)}\sqrt{d}} \cdot \frac{1}{s-1} + B(d) + \beta(d,G) + O(s-1),
$$

where $B(d)$ depends only on d and not on G (see (100)) *and*

$$\beta(d,G) = \frac{1}{2^{t(d)+1}} \sum_{\substack{d_1 \in F(d)\\ d_1 \notin \{1,\Delta\}}} \gamma_{d_1}(G) L(1,d_1) L(1,\Delta(d/d_1))$$
$$\times \sum_{m \mid f(d/d_1)} \frac{1}{m} \prod_{\substack{p \mid m\\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p}\right) p^{-1}\right) \prod_{p \mid f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-1}\right)$$
$$\times \prod_{p \mid f/m} \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-1}\right).$$

Proof. By Theorem 3, we have

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = S_1 + S_2, \tag{92}$$

where

$$S_1 = \frac{h(d)\log\varepsilon(d)}{2^{t(d)+1}} \sum_{m \mid f} \frac{1}{\log\varepsilon(d/m^2) h(d/m^2)} \cdot \frac{1}{m^{2s}} \tag{93}$$
$$\times 2\zeta(s) L(s,\Delta) \prod_{p \mid f/m} (1 - p^{-s}) \left(1 - \left(\frac{\Delta}{p}\right) p^{-s}\right),$$

and

$$S_2 = \frac{h(d)\log\varepsilon(d)}{2^{t(d)+1}} \sum_{m \mid f} \frac{1}{\log\varepsilon(d/m^2) h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{\substack{d_1 \in F(d/m^2)\\ d_1 \notin \{1,\Delta\}}} \gamma_{d_1}(G) \tag{94}$$
$$\times \prod_{p \mid f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-s}\right) \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-s}\right)$$
$$\times L(s,d_1) L(s,\Delta(d/d_1)).$$

We first deal with S_2. We have

$$S_2 = \frac{1}{2^{t(d)+1}} \sum_{m \mid f} \frac{h(d)\log\varepsilon(d)}{h(d/m^2)\log\varepsilon(d/m^2)} \cdot \frac{1}{m^2} \sum_{\substack{d_1 \in F(d/m^2)\\ d_1 \notin \{1,\Delta\}}} \gamma_{d_1}(G)$$
$$\times \prod_{p \mid f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-1}\right) \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-1}\right)$$
$$\times L(1,d_1) L(1,\Delta(d/d_1)) + O(s-1).$$

We recall (see for example [12: Theorem 11.2, p. 322]), that if l is a nonsquare discriminant and $d = lm^2$ then

$$\frac{L(1,d)}{L(1,l)} = \prod_{p|m}\left(1 - \left(\frac{l}{p}\right)p^{-1}\right). \tag{95}$$

By (95) and Dirichlet's class number formula (see for example [12: Theorem 10.1, p. 321]), we have

$$\begin{aligned}\frac{h(d)\log\varepsilon(d)}{h(d/m^2)\log\varepsilon(d/m^2)} &= \frac{\sqrt{d}\,L(1,d)}{\sqrt{d/m^2}\,L(1,d/m^2)} \\ &= m\prod_{p|m}\left(1 - \left(\frac{d/m^2}{p}\right)p^{-1}\right) \\ &= m\prod_{\substack{p|m \\ p\nmid f/m}}\left(1 - \left(\frac{\Delta}{p}\right)p^{-1}\right).\end{aligned} \tag{96}$$

Using (96), we obtain

$$\begin{aligned}S_2 = \frac{1}{2^{t(d)+1}} \sum_{m|f} \frac{1}{m} \prod_{\substack{p|m \\ p\nmid f/m}}\left(1 - \left(\frac{\Delta}{p}\right)p^{-1}\right) &\sum_{\substack{d_1\in F(d/m^2) \\ d_1\notin\{1,\Delta\}}} \gamma_{d_1}(G) \\ \times \prod_{p|f/m}\left(1 - \left(\frac{d_1}{p}\right)p^{-1}\right)&\left(1 - \left(\frac{\Delta(d/d_1)}{p}\right)p^{-1}\right) \\ \times L(1,d_1)L(1,\Delta(d/d_1)) &+ O(s-1).\end{aligned}$$

Interchanging the orders of summation and appealing to Lemma 1(e), we obtain

$$\begin{aligned}S_2 &= \frac{1}{2^{t(d)+1}} \sum_{\substack{d_1\in F(d) \\ d_1\notin\{1,\Delta\}}} \gamma_{d_1}(G)L(1,d_1)L(1,\Delta(d/d_1)) \sum_{m|f(d/d_1)} \frac{1}{m} \\ &\quad\times \prod_{\substack{p|m \\ p\nmid f/m}}\left(1 - \left(\frac{\Delta}{p}\right)p^{-1}\right) \prod_{p|f/m}\left(1 - \left(\frac{d_1}{p}\right)p^{-1}\right) \\ &\quad\times \prod_{p|f/m}\left(1 - \left(\frac{\Delta(d/d_1)}{p}\right)p^{-1}\right) + O(s-1) \\ &= \beta(d,G) + O(s-1).\end{aligned} \tag{97}$$

By (93) and (96), we have

$$S_1 = \frac{\zeta(s)}{2^{t(d)}}\,A(s,d),$$

where

$$A(s,d)=L(s,\Delta)\sum_{m\mid f}\frac{1}{m^{2s-1}}\prod_{\substack{p\mid m\\ p\nmid f/m}}\left(1-\left(\frac{\Delta}{p}\right)p^{-1}\right) \times\prod_{p\mid f/m}(1-p^{-s})\left(1-\left(\frac{\Delta}{p}\right)p^{-s}\right). \tag{98}$$

Hence, we obtain

$$S_1=\frac{A(1,d)}{2^{t(d)}}\cdot\frac{1}{s-1}+B(d)+O(s-1), \tag{99}$$

where

$$B(d)=(A'(1,d)+\gamma A(1,d))/2^{t(d)}. \tag{100}$$

We emphasize that $B(d)$ depends only on d and not on the genus G. By (95) and Dirichlet's class number formula, we obtain

$$\begin{aligned}L(1,\Delta)&=L(1,d)\prod_{p\mid f}\left(1-\left(\frac{\Delta}{p}\right)p^{-1}\right)^{-1}\\ &=\frac{h(d)\log\varepsilon(d)}{\sqrt{d}}\prod_{p\mid f}\left(1-\left(\frac{\Delta}{p}\right)p^{-1}\right)^{-1}.\end{aligned} \tag{101}$$

By (98) with $s=1$ and (101), we obtain after some simplification

$$A(1,d)=\frac{h(d)\log\varepsilon(d)}{\sqrt{d}}\sum_{m\mid f}\frac{1}{m}\prod_{p\mid f/m}(1-p^{-1})=\frac{h(d)\log\varepsilon(d)}{\sqrt{d}}. \tag{102}$$

By (99) and (102), we obtain

$$S_1=\frac{h(d)\log\varepsilon(d)}{2^{t(d)}\sqrt{d}}\cdot\frac{1}{s-1}+B(d)+O(s-1). \tag{103}$$

By (103), (97) and (92), we obtain the required result. ∎

5. Evaluation of some definite integrals. Proofs of Theorems 5–10. Theorem 5, which is a consequence of Theorems 2 and 4, evaluates a class of definite integrals. Theorems 6–10 all follow from Theorem 5.

THEOREM 5. *Let G_1, $G_2\in G(d)$. Then*

$$\int_0^1\Big(\sum_{[a,b,c]\in G_1}E(a,b,c,t)-\sum_{[a,b,c]\in G_2}E(a,b,c,t)\Big)\,dt$$

$$= \sum_{[a,b,c]\in G_1} \left(\frac{\pi^2\sqrt{d}}{6a} - J(a,b,c) + \log\varepsilon(d)\log a \right)$$

$$- \sum_{[a,b,c]\in G_2} \left(\frac{\pi^2\sqrt{d}}{6a} - J(a,b,c) + \log\varepsilon(d)\log a \right)$$

$$- \sqrt{d}\,(\beta(d,G_1) - \beta(d,G_2)),$$

where all forms (a, b, c) *are chosen so that* $a > 0$,

$$E(a,b,c,t) = \sum_{l=1}^{[(g-1)/2]} (F(\alpha,l,t,g) + F(\alpha',l,t,g)) + (1+(-1)^g)\frac{\log(1+t^{\varepsilon})}{1+t},$$

and

$$J(a,b,c)$$
$$= -2 \sum_{l=1}^{[(g-1)/2]} \left(\log(2\sin(\pi l/g))\log(2|\sin(\pi l\alpha/g)|) - \left(\frac{\pi}{2} - \frac{\pi l}{g}\right)\left(\frac{\pi}{2} - \frac{\pi l^*}{g}\right) \right)$$
$$- \frac{1+(-1)^g}{2}\log^2 2.$$

Proof. In this proof, all forms (a, b, c) satisfy $a > 0$. We also write $\varepsilon = \varepsilon(d)$. By Theorem 2, we have

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \sum_{[a,b,c]\in G} Z_{(a,b,c)}(s)$$
$$= \sum_{[a,b,c]\in G} \left(\frac{\log\varepsilon}{\sqrt{d}} \cdot \frac{1}{s-1} + V(d) + \frac{\pi^2}{6a} + \frac{\log\varepsilon\,\log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) + O(s-1)$$
$$= \frac{h(d)\log\varepsilon}{2^{t(d)}\sqrt{d}} \cdot \frac{1}{s-1} + \frac{V(d)h(d)}{2^{t(d)}} + \sum_{[a,b,c]\in G} \left(\frac{\pi^2}{6a} + \frac{\log a\,\log\varepsilon}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) + O(s-1).$$

By comparing with Theorem 4, we obtain

$$B(d) + \beta(d,G) = \frac{V(d)h(d)}{2^{t(d)}} + \sum_{[a,b,c]\in G} \left(\frac{\pi^2}{6a} + \frac{\log\varepsilon\,\log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right),$$

which is (15). Thus

$$\beta(d, G_1) - \beta(d, G_2) = \sum_{[a,b,c]\in G_1} \left(\frac{\pi^2}{6a} + \frac{\log\varepsilon\, \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) - \sum_{[a,b,c]\in G_2} \left(\frac{\pi^2}{6a} + \frac{\log\varepsilon\, \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right),$$

which is (16). The result follows on noting that

$$W_{(a,b,c)} = \int_0^1 E(a,b,c,t)\, dt + J(a,b,c)$$

and rearranging terms. ■

We now set

$$D = dy_0(d)^2. \tag{104}$$

Then $\varepsilon(D) = \varepsilon(d) = \varepsilon$ and $y_0(D) = 1$. Since $D+4$ is a square, we have $D+4 \equiv 0, 1, 4$ or $9 \pmod{16}$, so that $D \equiv 12, 13, 0$ or $5 \pmod{16}$. We are only interested in those D for which $H(D)$ contains a class of the type $[2, b, c]$ or $[4, b, c]$. This rules out $D \equiv 5 \pmod 8$, and so we are only interested in the cases $D \equiv 12 \pmod{16}$ and $D \equiv 0 \pmod{16}$. In the case $D \equiv 0 \pmod{16}$, we also have $\varepsilon(D/4) = \varepsilon$ and $y_0(D/4) = 2$.

If D is a positive integer such that $D \equiv 12 \pmod{16}$, $D+4$ is a square and $H(D)$ has one class per genus, then we show in Theorem 6 that we can explicitly evaluate $\int_0^1 \frac{\log(1+t^\varepsilon)}{1+t}\, dt$.

If D is a positive integer such that $D \equiv 0 \pmod{16}$, $D+4$ is a square, $D/4 \equiv 8 \pmod{16}$ and $H(D)$ has one class per genus, then we show in Theorems 8 and 9 that we can evaluate explicitly both of the integrals $\int_0^1 \frac{\log(1+t^\varepsilon)}{1+t}\, dt$ and $\int_0^1 \frac{\tan^{-1}(t^\varepsilon)}{1+t^2}\, dt$.

Before continuing we note the values of $E(a,b,c,t)$ and $J(a,b,c)$ for $g = 1, 2$ and 4, which we shall need later.

If $g = 1$, we have $E(a,b,c,t) = J(a,b,c) = 0$.

If $g = 2$, we have

$$E(a,b,c,t) = 2 \cdot \frac{\log(1+t^\varepsilon)}{1+t}, \qquad J(a,b,c) = -\log^2 2.$$

If $g = 4$, we have

$$E(a,b,c,t) = 2 \cdot \frac{t\log(1+t^{2\varepsilon}) - 2(-1)^{(\alpha-1)/2}\tan^{-1}(t^\varepsilon)}{1+t^2} + 2 \cdot \frac{\log(1+t^\varepsilon)}{1+t},$$

$$J(a,b,c) = -\frac{3}{2}\log^2 2 + (-1)^{(\alpha-1)/2}\frac{\pi^2}{8}.$$

The next result is a slight modification of a result of Chowla ([2], [4: p. 967]). It is useful in proving that certain form classes are not equal.

LEMMA 17. *Let k and m be integers with $k > 1$, m not a square and $-(2k-2) < m < 2k+2$. Then the equation*

$$x^2 - (k^2-1)y^2 = m \tag{105}$$

has no solution in positive integers x and y.

Proof. We suppose that (105) has a solution in positive integers x and y. Let (x_1, y_1) be the solution in positive integers to (105) for which y_1 is least. Let

$$x_2 = |kx_1 - (k^2-1)y_1|, \qquad y_2 = |x_1 - ky_1|.$$

Then $x_2^2 - (k^2-1)y_2^2 = x_1^2 - (k^2-1)y_1^2 = m$. If $y_2 = 0$ then $m = x_2^2$, a contradiction. Thus, $y_2 \geq 1$. If $x_2 = 0$, we have

$$m = -(k^2-1)y_2^2 \leq -(k^2-1) \leq -(2k-2),$$

a contradiction. Thus $x_2 > 0$. Hence, by the minimality of y_1, we have $y_2 \geq y_1$. Thus, either $x_1 - ky_1 \geq y_1$ or $x_1 - ky_1 \leq -y_1$. If $x_1 - ky_1 \geq y_1$, we have

$$m = x_1^2 - (k^2-1)y_1^2 \geq ((k+1)^2 - (k^2-1))y_1^2 = (2k+2)y_1^2 \geq 2k+2,$$

a contradiction. Similarly if $x_1 - ky_1 \leq -y_1$, we have $m \leq -(2k-2)$, which is a contradiction. ■

First we consider the case $D \equiv 12 \pmod{16}$. For a positive integer D, it is easily checked that $D \equiv 12 \pmod{16}$ with $D+4$ a square if and only if $D = 4(4l^2-1)$ for some positive integer l.

LEMMA 18. *Let $D = 4(4l^2-1)$ for some positive integer l. Then*

$$\left[1, 0, -\frac{D}{4}\right] \neq \left[2, 2, \frac{4-D}{8}\right]$$

in $H(D)$.

Proof. If $[1, 0, -D/4] = [2, 2, (4-D)/8]$, we have

$$2 = x^2 - \frac{D}{4}y^2 = x^2 - (4l^2-1)y^2 \tag{106}$$

for some positive integers x, y. But, by Lemma 17, equation (106) has no solution in positive integers since $2 < 2(2l) + 2$. ■

THEOREM 6. *Let $D = 4(4l^2-1)$ for some positive integer l and suppose that $H(D)$ has one class per genus. Let G_1 be the genus containing $[1, 0, -D/4]$ and let G_2 be the genus containing $[2, 2, (4-D)/8]$. Then*

$$\int_0^1 \frac{\log(1+t^\varepsilon)}{1+t}\,dt = \frac{\sqrt{D}}{2}\left(\beta(D, G_1) - \beta(D, G_2)\right) - \frac{\pi^2\sqrt{D}}{24} + \frac{\log 2 \, \log 2\varepsilon}{2},$$

where $\varepsilon = \varepsilon(D) = 2l + \sqrt{4l^2-1}$.

Proof. We observe that $G_1 \neq G_2$ by Lemma 18. The result follows on using Theorem 5, noting that $y_0(D) = 1$, $g = 1$ for the form $(1, 0, -D/4)$, and $g = 2$ for the form $(2, 2, (4-D)/8)$ and using the values of $E(a, b, c, t)$ and $J(a, b, c)$ given just before Lemma 17. ∎

The following are the first few cases where the conditions of Theorem 6 are satisfied so that we can calculate $\int_0^1 \frac{\log(1+t^\varepsilon)}{1+t}\,dt$:

- $l = 1$, $D = 12$, $\varepsilon = 2 + \sqrt{3}$,
- $l = 2$, $D = 60$, $\varepsilon = 4 + \sqrt{15}$,
- $l = 3$, $D = 140$, $\varepsilon = 6 + \sqrt{35}$,
- $l = 4$, $D = 252$, $\varepsilon = 8 + \sqrt{63}$,
- $l = 6$, $D = 572$, $\varepsilon = 12 + \sqrt{143}$,
- $l = 7$, $D = 780$, $\varepsilon = 14 + \sqrt{195}$.

Theorem 7.

$$\int_0^1 \frac{\log(1+t^{2+\sqrt{3}})}{1+t}\,dt = \frac{\pi^2}{12}(1-\sqrt{3}) + \log 2\,\log(1+\sqrt{3}).$$

$$\int_0^1 \frac{\log(1+t^{4+\sqrt{15}})}{1+t}\,dt = \frac{\pi^2}{12}(2-\sqrt{15}) + \log\left(\frac{1+\sqrt{5}}{2}\right)\log(2+\sqrt{3}) + \log 2\,\log(\sqrt{3}+\sqrt{5}).$$

$$\int_0^1 \frac{\log(1+t^{6+\sqrt{35}})}{1+t}\,dt = \frac{\pi^2}{12}(3-\sqrt{35}) + \log\left(\frac{1+\sqrt{5}}{2}\right)\log(8+3\sqrt{7}) + \log 2\,\log(\sqrt{5}+\sqrt{7}).$$

$$\int_0^1 \frac{\log(1+t^{8+\sqrt{63}})}{1+t}\,dt = \frac{\pi^2}{12}(4-\sqrt{63}) + \log\left(\frac{5+\sqrt{21}}{2}\right)\log(2+\sqrt{3}) + \log 2\,\log(3+\sqrt{7}).$$

$$\int_0^1 \frac{\log(1+t^{12+\sqrt{143}})}{1+t}\,dt = \frac{\pi^2}{12}(6-\sqrt{143}) + \log\left(\frac{3+\sqrt{13}}{2}\right)\log(10+3\sqrt{11}) + \log 2\,\log(\sqrt{11}+\sqrt{13}).$$

$$\int_0^1 \frac{\log(1+t^{14+\sqrt{195}})}{1+t}\,dt = \frac{\pi^2}{12}(7-\sqrt{195}) + \log\left(\frac{1+\sqrt{5}}{2}\right)\log(25+4\sqrt{39}) + \log\left(\frac{3+\sqrt{13}}{2}\right)\log(4+\sqrt{15}) + \log 2\,\log(\sqrt{15}+\sqrt{13}).$$

The second, third and fifth integrals in Theorem 7 are due to Herglotz [11, p. 14].

We now turn to the case $D \equiv 0 \pmod{16}$. Let D be a positive integer. Then $D \equiv 0 \pmod{16}$ with $D+4$ a square if and only if $D = 16(l^2+l)$ for some positive integer l. If D has this form, then $D \equiv 0 \pmod{32}$ and

$$\frac{D}{4} \equiv \begin{cases} 0 \pmod{16} & \text{if } l \equiv 0 \text{ or } 3 \pmod 4, \\ 8 \pmod{16} & \text{if } l \equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

LEMMA 19. *Let $D = 16(l^2+l)$ for some positive integer l with $l \equiv 1$ or $2 \pmod 4$. Then $H(D)$ and $H(D/4)$ have the same number of classes per genus.*

Proof. Since $D+4$ is a square, we have $\varepsilon(D) = \varepsilon(D/4)$. Hence (96) gives

$$\frac{h(D)}{h(D/4)} = 2.$$

Since $D \equiv 0 \pmod{32}$, we have $t(D) = \omega(D)$. Since $D/4 \equiv 8 \pmod{16}$, we have $t(D/4) = \omega(D/4) - 1 = \omega(D) - 1$. Hence $t(D) = 1 + t(D/4)$. Thus

$$\frac{h(D/4)}{2^{t(D/4)}} = \frac{h(D)/2}{2^{t(D/4)}} = \frac{h(D)}{2^{t(D)}}$$

as required. ■

LEMMA 20. *Let $D = 16(l^2+l)$ for some positive integer l. Then*

$$[1, 0, -D/4] \neq [4, 4, (16-D)/16] \quad \text{in } H(D).$$

Proof. Suppose that $[1, 0, -D/4] = [4, 4, (16-D)/16]$. Then there exist coprime integers α, γ such that $\alpha^2 - D\gamma^2/4 = 4$. Thus

$$\frac{\alpha + \gamma\sqrt{D/4}}{2} = \pm\varepsilon(D/4)^n,$$

for some integer n. But this gives

$$\alpha + \gamma\sqrt{D/4} = \pm 2(2l+1+\sqrt{D/4})^n,$$

so that α and γ are even, a contradiction. ■

LEMMA 21. *Let $D = 16(l^2+l)$ for some positive integer l with $l \equiv 1$ or $2 \pmod 4$. Then $[1, 0, -D/16] \neq [2, 0, -D/32]$ in $H(D/4)$ except if $l = 1$.*

Proof. If $l = 1$, we have $[1, 0, -2] = [2, 0, -1]$ in $H(8)$. If $l > 1$ and $[1, 0, -D/16] = [2, 0, -D/32]$ in $H(D/4)$, then there exist positive integers x, y with $x^2 - Dy^2/16 = 2$. Hence

$$8 = (2x)^2 - \frac{D}{4}y^2 = u^2 - ((2l+1)^2 - 1)v^2 \tag{107}$$

for some positive integers u, v. But, by Lemma 17, the equation (107) has no solution in positive integers since $8 < 2(2l+1)+2$. ■

THEOREM 8. *Let $D = 16(l^2+l)$ for some positive integer l. Suppose that $H(D)$ has one class per genus. Let G_1 be the genus containing $[1, 0, -D/4]$ and let G_2 be the genus containing $[4, 4, (16-D)/16]$. Then*

$$\int_0^1 \frac{3t\log(1+t^{2\varepsilon}) + 2(-1)^l \tan^{-1}(t^{\varepsilon})}{1+t^2}\,dt$$
$$= \frac{\sqrt{D}}{2}\,(\beta(D,G_1) - \beta(D,G_2)) - \frac{\pi^2\sqrt{D}}{16} + (-1)^l\,\frac{\pi^2}{16} + \log 2\,\log(2^{3/4}\varepsilon),$$

where $\varepsilon = \varepsilon(D) = 2l+1+\sqrt{D/4} = 2l+1+\sqrt{4l^2+4l}$.

Proof. From Lemma 20 we see that $G_1 \neq G_2$. We have $y_0(D) = 1$. For the form $(1, 0, -D/4)$, we have $g = 1$. For the form $(4, 4, (16-D)/16)$, we have $g = 4$, $\alpha = 2l-1$. Using these facts together with the values of $E(a, b, c, t)$ and $J(a, b, c)$ given before Lemma 17 and the relation

$$\int_0^1 \frac{\log(1+t^{\varepsilon})}{1+t}\,dt = 2\int_0^1 \frac{t\log(1+t^{2\varepsilon})}{1+t^2}\,dt$$

in Theorem 5 gives the required result. ■

In a similar manner, we obtain

THEOREM 9. *Let $D = 16(l^2+l)$ for some positive integer l with $l \equiv 1$ or $2 \pmod 4$ (so that $D/4 \equiv 8 \pmod{16}$). Let $H(D)$ have one class per genus so that $H(D/4)$ also has one class per genus by Lemma 19. In $H(D/4)$, let $\widehat{G}_1$ be the genus containing $[1, 0, -D/16]$ and let $\widehat{G}_2$ be the genus containing $[2, 0, -D/32]$. Then*

$$\int_0^1 \frac{t\log(1+t^{2\varepsilon}) + 2(-1)^{l+1} \tan^{-1}(t^{\varepsilon})}{1+t^2}\,dt$$
$$= \frac{\sqrt{D}}{4}\,(\beta(D/4,\widehat{G}_1) - \beta(D/4,\widehat{G}_2)) - \frac{\pi^2\sqrt{D}}{48} + (-1)^{l+1}\,\frac{\pi^2}{16} + \frac{\log 2\,\log(\sqrt{2}\,\varepsilon)}{2},$$

where $\varepsilon = \varepsilon(D) = \varepsilon(D/4) = 2l+1+\sqrt{4l^2+4l}$.

We note by Lemma 21 that $\widehat{G}_1 \neq \widehat{G}_2$ if $l \neq 1$. If $D = 16(l^2+l)$, for some positive integer l with $l \equiv 1$ or $2 \pmod 4$ and $H(D)$ has one class per genus, both Theorems 8 and 9 are applicable. Thus we can calculate both

$$\int_0^1 \frac{\log(1+t^{\varepsilon})}{1+t}\,dt = 2\int_0^1 \frac{t\log(1+t^{2\varepsilon})}{1+t^2}\,dt \quad\text{and}\quad \int_0^1 \frac{\tan^{-1}(t^{\varepsilon})}{1+t^2}\,dt.$$

The following are the first few cases where this happens:

- $l = 1,\ D = 32,\ \varepsilon = 3 + \sqrt{8}$,
- $l = 2,\ D = 96,\ \varepsilon = 5 + \sqrt{24}$,
- $l = 5,\ D = 480,\ \varepsilon = 11 + \sqrt{120}$,
- $l = 6,\ D = 672,\ \varepsilon = 13 + \sqrt{168}$.

Applying Theorems 8 and 9 in these cases, we obtain

THEOREM 10.

$$\int_0^1 \frac{\log(1+t^{3+\sqrt{8}})}{1+t}\,dt = \frac{\pi^2}{24}(3-\sqrt{32}) + \frac{1}{2}\log 2\,\log(2(3+\sqrt{8})^{3/2}),$$

$$\int_0^1 \frac{\tan^{-1}(t^{3+\sqrt{8}})}{1+t^2}\,dt = \frac{1}{16}\log 2\,\log(3+\sqrt{8}).$$

$$\begin{aligned}\int_0^1 \frac{\log(1+t^{5+\sqrt{24}})}{1+t}\,dt &= \frac{\pi^2}{24}(5-\sqrt{96}) + \frac{1}{2}\log(1+\sqrt{2})\log(2+\sqrt{3})\\ &\quad + \frac{1}{2}\log 2\,\log(2(5+\sqrt{24})^{3/2}),\end{aligned}$$

$$\int_0^1 \frac{\tan^{-1}(t^{5+\sqrt{24}})}{1+t^2}\,dt = \frac{1}{8}\log(1+\sqrt{2})\log(2+\sqrt{3}) - \frac{1}{16}\log 2\,\log(5+\sqrt{24}).$$

$$\begin{aligned}\int_0^1 \frac{\log(1+t^{11+\sqrt{120}})}{1+t}\,dt &= \frac{\pi^2}{24}(11-\sqrt{480}) + \frac{1}{2}\log(1+\sqrt{2})\log(4+\sqrt{15})\\ &\quad + \frac{1}{2}\log(2+\sqrt{3})\log(3+\sqrt{10})\\ &\quad + \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right)\log(5+\sqrt{24})\\ &\quad + \frac{1}{2}\log 2\,\log(2(11+\sqrt{120})^{3/2}),\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{\tan^{-1}(t^{11+\sqrt{120}})}{1+t^2}\,dt &= -\frac{1}{8}\log(1+\sqrt{2})\log(4+\sqrt{15})\\ &\quad - \frac{1}{8}\log(2+\sqrt{3})\log(3+\sqrt{10})\\ &\quad + \frac{3}{8}\log\left(\frac{1+\sqrt{5}}{2}\right)\log(5+\sqrt{24})\\ &\quad + \frac{1}{16}\log 2\,\log(11+\sqrt{120}).\end{aligned}$$

$$\int_0^1 \frac{\log(1+t^{13+\sqrt{168}})}{1+t}\,dt = \frac{\pi^2}{24}(13-\sqrt{672})$$
$$+\frac{1}{2}\log(1+\sqrt{2})\log\left(\frac{5+\sqrt{21}}{2}\right)$$
$$+\frac{1}{4}\log(2+\sqrt{3})\log(15+\sqrt{224})$$
$$+\frac{1}{4}\log(5+\sqrt{24})\log(8+\sqrt{63})$$
$$+\frac{1}{2}\log 2\ \log(2(13+\sqrt{168})^{3/2}),$$

$$\int_0^1 \frac{\tan^{-1}(t^{13+\sqrt{168}})}{1+t^2}\,dt = -\frac{3}{8}\log(1+\sqrt{2})\log\left(\frac{5+\sqrt{21}}{2}\right)$$
$$+\frac{1}{16}\log(2+\sqrt{3})\log(15+\sqrt{224})$$
$$+\frac{1}{16}\log(5+\sqrt{24})\log(8+\sqrt{63})$$
$$-\frac{1}{16}\log 2\ \log(13+\sqrt{168}).$$

References

[1] D. A. Buell, *Binary Quadratic Forms*, Springer, New York, 1989.

[2] S. Chowla, *On the inequality* $|x^2-y^2-2xyk| \geq 2k$ $(x, y, k$ *odd*$)$, Norske Vid. Selsk. Forh. (Trondheim) 34 (1961), 91. [Chowla's Collected Papers, Vol. II, p. 967.]

[3] —, *Remarks on class-invariants and related topics*, in: 1963 Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), Part II, Calcutta Math. Soc., Calcutta, 361–372. [Chowla's Collected Papers, Vol. III, 1008–1019.]

[4] —, *Collected Papers* (3 Volumes), ed. by J. G. Huard and K. S. Williams, Centre de Recherches Math., Univ. Montréal, 1999.

[5] S. Chowla and A. Selberg, *On Epstein's zeta function (I)*, Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 371–374. [Chowla's Collected Papers, Vol. II, 719–722.]

[6] C. Deninger, *On the analogue of the formula of Chowla and Selberg for real quadratic fields*, J. Reine Angew. Math. 351 (1984), 171–191.

[7] P. G. L. Dirichlet, *Vorlesungen über Zahlentheorie*, Chelsea, New York, 1968.

[8] P. Epstein, *Zur Theorie allgemeiner Zetafunctionen*, Math. Ann. 56 (1903), 615–644.

[9] D. R. Estes and G. Pall, *Spinor genera of binary quadratic forms*, J. Number Theory 5 (1973), 421–432.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals*, *Series*, *and Products*, 5th ed., Academic Press, 1994.

[11] G. Herglotz, *Über die Kroneckersche Grenzformel für reele*, *quadratische Körper. I.*, Ber. d. Sacks. Akad. d. Wiss. zu Leipzig 75 (1923), 3–14.

[12] L.-K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982.

[13] J. G. Huard, P. Kaplan and K. S. Williams, *The Chowla–Selberg formula for genera*, Acta Arith. 73 (1995), 271–301.
[14] A. Selberg and S. Chowla, *On Epstein's zeta-function*, J. Reine Angew. Math. 227 (1967), 86–110. [Chowla's Collected Papers, Vol. III, 1101–1125.]
[15] C. L. Siegel, *Advanced Analytic Number Theory*, Tata Inst. Fund. Research, Bombay, 1980.
[16] D. Zagier, *A Kronecker limit formula for real quadratic fields*, Math. Ann. 213 (1975), 153–184.

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